

1 Problems for all Students

Problem 1. Let X be a standard Gaussian random variable. Show that $\text{Cov}(X, X^2) = 0$, yet X and X^2 are not independent.

Problem 2. Let X be a standard Gaussian random variable and define the random variable Y by

$$Y = \begin{cases} X & \text{if } |X| \leq 1, \\ -X & \text{if } |X| > 1. \end{cases}$$

- (a) Show that Y is also a standard Gaussian random variable.
- (b) Argue that $X + Y$ is not a Gaussian random variable.
- (c) Explain why the random vector $\vec{X} = (X, Y)$ is not a Gaussian random vector.

Remark: This is an important example to keep in mind. It explains why we cannot define a Gaussian random vector by simply stating that each of its components is Gaussian.

Problem 3. Let C be the covariance matrix of the random vector $(X_1, \dots, X_n)$ (not necessarily Gaussian). Show that C is always positive semidefinite, i.e.,

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j C_{ij} \geq 0,$$

for all $a_1, \dots, a_n \in \mathbb{R}$. **Hint:** Write the left side as the variance of some random variable.

Problem 4. Let $\vec{X} = (X_1, \dots, X_n)$ be an n -dimensional Gaussian vector and M an $m \times n$ matrix.

- (a) Show that $\vec{Y} = M\vec{X}$ is also a Gaussian vector.
- (b) If the covariance matrix of $\vec{X}$ is C , write the covariance matrix of $\vec{Y}$ in terms of M and C .

Problem 5. Let $\vec{X} = (X, Y)$ be a Gaussian random vector with mean 0 and covariance matrix

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

for $\rho \in (-1, 1)$. Write (X, Y) as a linear combination of IID standard Gaussian random variables. Write down the PDF of the Gaussian vector.

Problem 6. Consider the Gaussian vector $\vec{X} = (X_1, X_2, X_3)$ with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

If $\vec{Z} = (Z_1, Z_2, Z_3)$, with Z_1, Z_2, Z_3 IID standard Gaussian random variables, find a matrix A such that $\vec{X} = A\vec{Z}$.

2 Problems for Graduate Students

Problem 7. Let $\vec{X} = (X_1, \dots, X_n)$ be a random vector with covariance matrix C .

- (a) Show that C is symmetric, i.e., $C^T = C$.
- (b) Let λ_1 and λ_2 be distinct eigenvalues of C , with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Show that $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \vec{v}_2 = 0$. **Hint:** Compute $\vec{v}_1^T C \vec{v}_2$ in two different ways.
- (c) Suppose λ is an eigenvalue of C with corresponding eigenvector $\vec{v} \neq \vec{0}$. Using the fact that C is positive semidefinite, show that $\lambda \geq 0$.
- (d) Recall that a real-valued $n \times n$ matrix Q is called *unitary* if its columns form an orthonormal basis of $\mathbb{R}^n$. Show that if Q is unitary, then $Q^{-1} = Q^T$.
- (e) Explain why for any covariance matrix C there exists a unitary matrix Q and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ such that $C = Q\Lambda Q^T$.
- (f) Show that there exists a matrix A such that $C = AA^T$.

Problem 8. Let $\vec{X} = (X_1, \dots, X_n)$ be a Gaussian random vector with mean 0 and covariance matrix C . If $\vec{Z} = (Z_1, \dots, Z_n)$, where Z_i are IID standard Gaussian random variables, show that there exists a matrix A such that $\vec{X} = A\vec{Z}$.

Homework #4

#1

Let X be a standard Gaussian random variable. Show that $\text{Cov}(X, X^2) = 0$, yet X and X^2 are not independent.

Solution:

$\text{Cov}(X, X^2) = \mathbb{E}[X^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx = 0$. However, if we assume X, X^2 are independent we obtain the following contradiction:

$$0 = P(1 < X < 2, 9 < X^2 < 16) = P(1 < X < 2)P(9 < X^2 < 16) > 0.$$

#2

Let X be a standard Gaussian random variable and define

$$Y = \begin{cases} X & \text{if } |X| \leq 1 \\ -X & \text{if } |X| > 1 \end{cases}$$

(a) Show that Y is also a standard Gaussian.

(b) Argue that $X+Y$ is not a Gaussian random variable.

(c) Explain why $\vec{X} = (X, Y)$ is not a Gaussian random vector.

Solution:

(a) If we let $F_X(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$ then

$$F_Y(y) = \begin{cases} P(-X \leq y) & \text{if } y < -1 \\ P(-X < 1) + P(-1 \leq X \leq y) & \text{if } -1 \leq y \leq 1 \\ P(-X < 1) + P(-1 \leq X \leq 1) + P(1 \leq X \leq y) & \text{if } 1 < y \end{cases}$$

$$\Rightarrow F_X(y) = \begin{cases} P(X > -y) & \text{if } y < -1 \\ P(X > 1) + P(-1 \leq X \leq y) & \text{if } -1 \leq y \leq 1 \\ P(X > 1) + P(-1 \leq X \leq 1) + P(1 \leq X \leq -y) & \text{if } 1 < y \end{cases}$$

Now, by symmetry of the Gaussian density we have that

$$P(X > -y) = P(X < y) \text{ if } y < 0$$

$$P(X > 1) = P(X < -1)$$

$$P(-y \leq X \leq -1) = P(1 \leq X \leq y) \text{ if } y > 1$$

Therefore,

$$F_X(y) = \begin{cases} P(X < y), & y < -1 \\ P(X < -1) + P(-1 \leq X \leq y), & -1 \leq y \leq 1 \\ P(X < -1) + P(-1 \leq X \leq 1) + P(1 \leq X \leq y), & y > 1 \end{cases}$$

$$\Rightarrow F_X(y) = \begin{cases} P(X < y), & y < -1 \\ P(X < y), & -1 \leq y \leq 1 \\ P(X < y), & y > 1 \end{cases}$$

$$\Rightarrow F_X(y) = F_X(y)$$

and thus Y is Gaussian.

b) $X+Y$ is not Gaussian since $P(X+Y > 1) = 0$.

c) $\vec{X} = (X, Y)$ is not a Gaussian random variable since there exists a linear combination of X and Y , namely $X+Y$, which is not Gaussian.

3

Let C be the covariance matrix of the random vector $(X_1, \dots, X_n)$. Show that C is positive definite.

Proof.

For $a_1, \dots, a_n \in \mathbb{R}$ let $Y = a_1 X_1 + \dots + a_n X_n$. Consequently, if we let $\vec{a}$ be the mean vector for this random vector we have that

$$\begin{aligned} E(Y) &= \vec{a}^T \vec{\mu} = \vec{a} \cdot \vec{\mu} \\ \sum_{i=1}^n \sum_{j=1}^n a_i a_j C_{ij} &= \vec{a}^T C \vec{a} = E[(Y - \vec{a} \cdot \vec{\mu})^2] \geq 0. \end{aligned}$$

#4

Let $\vec{X} = (X_1, \dots, X_n)$ be an n -dimensional Gaussian vector and M an $m \times n$ matrix.

(a) Show that $\vec{Y} = M\vec{X}$ is also a Gaussian vector.

(b) If the covariance matrix of $\vec{X}$ is C , write the covariance matrix of $\vec{Y}$ in terms of M and C .

Solution:

(a) Let $\vec{Y} = (Y_1, \dots, Y_m)$ be defined by $\vec{Y} = M\vec{X}$. Therefore, for $b_1, \dots, b_m \in \mathbb{R}$ we have that

$$\begin{aligned} b_1 Y_1 + \dots + b_m Y_m &= b_1 \sum_{i=1}^n M_{1i} X_i + \dots + b_m \sum_{i=1}^n M_{mi} X_i \\ &= \sum_{i=1}^n \sum_{j=1}^m b_j M_{ji} X_i \\ &= \sum_{i=1}^n \sum_{j=1}^m b_j M_{ji} X_i \\ &= \sum_{i=1}^n a_i X_i, \end{aligned}$$

where $a_i = \sum_{j=1}^m b_j M_{ji}$. Consequently, $b_1 Y_1 + \dots + b_m Y_m$ is a Gaussian random variable. Since $b_1, \dots, b_m \in \mathbb{R}$ were arbitrary it follows that $\vec{Y}$ is a Gaussian random vector.

#5

Let $\vec{X} = (X, Y)$ be a Gaussian random vector with $\vec{\mu} = 0$ and covariance matrix

$$C = \begin{bmatrix} 1 & g \\ g & 1 \end{bmatrix}$$

for $g \in (-1, 1)$. Write (X, Y) as a linear combination of IID standard Gaussian random variables. Write down the PDF of the Gaussian vector.

Solution:

Let $Z_1 = X_1$. By construction $E[Z_1^2] = E[X_1^2] = 1$. Now, if we let

$$\begin{aligned} Z_2' &= X_2 - \langle Z_1, X_2 \rangle Z_1 \\ &= X_2 - \langle X_1, X_2 \rangle Z_1 \\ &= X_2 - \rho X_1 \end{aligned}$$

Furthermore,

$$\begin{aligned} E[Z_2'^2] &= E[(X_2 - \rho X_1)^2] \\ &= E[X_2^2] - 2\rho E[X_1 X_2] + \rho^2 E[X_1^2] \\ &= 1 - 2\rho^2 + \rho^2 \\ &= 1 - \rho^2. \end{aligned}$$

Therefore, we define

$$Z_2 = \frac{X_2 - \rho X_1}{\sqrt{1 - \rho^2}}$$

Consequently, we have that

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho/\sqrt{1-\rho^2} & -\rho/\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A \vec{X}$$

Therefore,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = A^{-1} \vec{X} = \frac{1}{\rho} \begin{bmatrix} -\rho/\sqrt{1-\rho^2} & 0 \\ -1/\sqrt{1-\rho^2} & 1 \end{bmatrix} \vec{X} = \begin{bmatrix} 1 & 0 \\ \rho & -\sqrt{1-\rho^2}/\rho \end{bmatrix} \vec{X}$$

6

Consider the Gaussian vector $\vec{X} = (X_1, X_2, X_3)$ with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

f $\vec{Z} = (Z_1, Z_2, Z_3)$, with Z_1, Z_2, Z_3 IID standard Gaussian random variables, and a matrix A such that $\vec{X} = A \vec{Z}$.

Solution:

1. Set $z_1 = \frac{1}{\sqrt{2}} X_1$

2. Set $z'_1 = X_2 - \langle X_2, z_1 \rangle z_1$. Calculating we have that

$$\langle X_2, z_1 \rangle = \langle X_2, \frac{1}{\sqrt{2}} X_1 \rangle = \frac{1}{\sqrt{2}}$$

Consequently,

$$z'_1 = X_2 - \frac{1}{2} X_1$$

$$\begin{aligned} \Rightarrow \langle z'_1, z'_1 \rangle &= \langle X_2, X_2 \rangle - \langle X_2, X_1 \rangle + \frac{1}{4} \langle X_1, X_1 \rangle \\ &= 2 - 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Therefore,

$$z_1 = \sqrt{\frac{2}{3}} z'_1 = -\frac{1}{\sqrt{6}} X_1 + \sqrt{\frac{2}{3}} X_2$$

3. Set $z'_2 = X_3 - \langle X_3, z_1 \rangle z_1 - \langle X_3, z'_1 \rangle z'_1$. Calculating, we have that

$$\langle X_3, z_1 \rangle = \langle X_3, \frac{1}{\sqrt{2}} X_1 \rangle = \frac{1}{\sqrt{2}}$$

$$\langle X_3, z'_1 \rangle = \langle X_3, -\frac{1}{\sqrt{6}} X_1 + \sqrt{\frac{2}{3}} X_2 \rangle = -\frac{1}{\sqrt{6}} + \sqrt{\frac{2}{3}} = \frac{1}{\sqrt{6}}$$

Consequently,

$$z'_2 = X_3 - \frac{1}{\sqrt{2}} z_1 - \frac{1}{\sqrt{6}} z'_1 = X_3 - \frac{1}{2} X_1 + \frac{1}{6} X_1 - \frac{1}{3} X_2$$

$$\Rightarrow z'_2 = X_3 - \frac{1}{3} X_1 - \frac{1}{3} X_2$$

$$\begin{aligned} \Rightarrow \langle z'_2, z'_2 \rangle &= \langle X_3, X_3 \rangle - \frac{2}{3} \langle X_3, X_1 \rangle - \frac{2}{3} \langle X_3, X_2 \rangle + \frac{1}{9} \langle X_1, X_1 \rangle \\ &\quad + \frac{2}{9} \langle X_1, X_2 \rangle + \frac{1}{9} \langle X_2, X_2 \rangle \\ &= 2 - \frac{2}{3} - \frac{2}{3} + \frac{1}{9} + \frac{2}{9} + \frac{1}{9} \\ &= 2 - \frac{2}{3} \\ &= \frac{4}{3} \end{aligned}$$

Therefore,

$$z_2 = \sqrt{\frac{3}{4}} z'_2 = \frac{\sqrt{3}}{2} X_3 - \frac{1}{\sqrt{6}} X_1 - \frac{1}{\sqrt{6}} X_2$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{3}{4}} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = A^{-1} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

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Let $\vec{X} = (X_1, \dots, X_n)$ be a random vector with covariance matrix C .

- Show that C is symmetric.
- Let λ_1, λ_2 be distinct eigenvalues of C , with eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Show that $\vec{v}_1 \cdot \vec{v}_2 = 0$.
- Suppose λ is an eigenvalue of C with corresponding eigenvector $\vec{v} \neq 0$. Using the fact that C is positive semidefinite, show that $\lambda \geq 0$.
- Show that a matrix is unitary if $Q^{-1} = Q^T$.
- Explain why for any covariance matrix C there exists a unitary matrix Q and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ such that $C = Q\Lambda Q^T$.
- Show that there exists a matrix A such that $C = AA^T$.

Solution:

$$(a) C_{ij} = \text{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] = \mathbb{E}[(X_j - \mu_j)(X_i - \mu_i)] = \text{Cov}(X_j, X_i) = C_{ji}.$$

(b) Let λ_1, λ_2 be distinct eigenvalues of C with corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. Therefore, by symmetry of C :

$$\begin{aligned} \vec{v}_1 \cdot C\vec{v}_2 &= \vec{v}_2 \cdot C\vec{v}_1 \\ \Rightarrow \vec{v}_1 \cdot (\lambda_2 \vec{v}_2) &= \vec{v}_2 \cdot (\lambda_1 \vec{v}_1) \\ \Rightarrow \lambda_2 \vec{v}_1 \cdot \vec{v}_2 &= \lambda_1 \vec{v}_2 \cdot \vec{v}_1 \\ \Rightarrow (\lambda_2 - \lambda_1) \vec{v}_1 \cdot \vec{v}_2 &= 0. \end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ it follows that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

(c) Suppose λ is an eigenvalue of C with corresponding eigenvector $\vec{v}$.

Therefore, since C is positive semidefinite it follows that

$$\vec{v} \cdot C\vec{v} = \vec{v} \cdot (\lambda \vec{v}) = \lambda \|\vec{v}\|^2 \geq 0$$

which implies $\lambda \geq 0$.

(d) If Q is unitary with columns $\vec{q}_1, \dots, \vec{q}_n$ then $\vec{q}_1, \dots, \vec{q}_n$ forms an orthonormal basis. Consequently, the ij entry of QQ^T is given by

$$(QQ^T)_{ij} = \vec{q}_i \cdot \vec{q}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Therefore, $Q^{-1} = Q^T$.

(e) This follows from the spectral theorem. That is if we let Q be a matrix whose columns consist of the n orthonormal distinct eigenvectors of C . Consequently,

$$\begin{aligned} CQ &= Q\Lambda \\ \Rightarrow C &= Q\Lambda Q^T. \end{aligned}$$

(f) By construction

$$\begin{aligned} C &= Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T = Q\sqrt{\Lambda}(Q\sqrt{\Lambda})^T \\ \text{and thus } A &= Q\sqrt{\Lambda}. \end{aligned}$$

#8.

Let $\vec{X}$ be a Gaussian random vector with mean 0 and covariance matrix C . If $\vec{Z} = (Z_1, \dots, Z_n)$, where Z_i are IID standard Gaussian random variables, show that there exists a matrix A such that $\vec{X} = A\vec{Z}$.

Solution:

If let $A = Q\sqrt{\Lambda}$, where Q is the matrix whose columns are the orthonormal eigenvectors of C with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ and $\sqrt{\Lambda} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. If we let $\vec{X} = A\vec{Z}$ then

$$\begin{aligned} \text{Cov}(\vec{X}, \vec{X}) &= \text{Cov}\left(\sum_{k=1}^n A_{ik} Z_k, \sum_{l=1}^n A_{jl} Z_l\right) \\ &= \sum_{k=1}^n \sum_{l=1}^n A_{ik} A_{jl} \text{Cov}(Z_k, Z_l) \\ &= \sum_{k=1}^n A_{ik} A_{jk} \\ &= (A \cdot A^T)_{ij} \\ &= Q\sqrt{\Lambda}(Q\sqrt{\Lambda})^T \\ &= Q\Lambda Q^T \\ &= C \end{aligned}$$