

## 1 Problems for all Students

**Problem 1.** Let  $B_t$  be a standard Brownian motion. If  $0 \leq s \leq t$ , use the properties of Brownian motion to compute each of the following:

- (a)  $\text{Cov}(B_s, B_t - B_s)$ .
- (b) The joint distribution of  $(B_s, B_t - B_s)$ .

**Problem 2.** Let  $B_t$  be a standard Brownian motion and let

$$X_t = \mu t + \sigma B_t,$$

where  $\mu \in \mathbb{R}$  and  $\sigma > 0$ .

- (a) Compute  $\mathbb{E}[X_t]$ ,  $\text{Var}(X_t)$ , and  $\text{Cov}(X_s, X_t)$  for  $0 \leq s \leq t$ .
- (b) For  $0 \leq s < t$ , determine the distribution of the increment  $X_t - X_s$ .
- (c) For  $0 \leq t_0 < t_1 < t_2 < t_3$ , determine whether the increments  $X_{t_1} - X_{t_0}$  and  $X_{t_3} - X_{t_2}$  are independent. Justify your answer.
- (d) Write down the probability density of  $X_t$  for  $t > 0$ .

**Problem 3.** Let  $B_t$  be a standard Brownian motion. Compute the following moments:

- (a)  $\mathbb{E}[B_t^4]$ .
- (b)  $\mathbb{E}[B_s B_t]$  if  $0 < s < t$ .
- (c)  $\mathbb{E}[B_s^2 B_t^2]$  if  $0 < s < t$ .
- (d)  $\mathbb{E}[B_s B_t^3]$  if  $0 < s < t$ .
- (e)  $\mathbb{E}[(B_t - B_s)^4]$  if  $0 < s < t$ .

**Hint:** When  $s < t$ , write  $B_t = B_s + (B_t - B_s)$  and use independence of Brownian increments.

**Problem 4.** Let  $B_t$  be a standard Brownian motion. Write each of the following probabilities as an iterated integral involving standard Gaussian densities. You do not need to evaluate the integrals.

- (a)  $P(B_1 > 1, B_2 > 1)$ .
- (b)  $P(B_1 > 0, B_2 > 1, B_3 > 2)$ .

**Hint:** Use the independent increments  $B_1$ ,  $B_2 - B_1$ , and  $B_3 - B_2$ .

## 2 Problems for Graduate Students

**Problem 7.** Let  $X$  and  $Y$  be random variables satisfying  $\mathbb{E}[X^2] < \infty$  and  $\mathbb{E}[Y^2] < \infty$ . Prove the Cauchy–Schwarz inequality

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]} \sqrt{\mathbb{E}[Y^2]}.$$

**Hint:** For any  $a \in \mathbb{R}$ , use the fact that

$$\mathbb{E}[(X - aY)^2] \geq 0.$$

### Definition: Convergence in $L^2$

A sequence of random variables  $X_1, X_2, \dots$  converges to a random variable  $X$  in  $L^2$  if

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|^2] = 0.$$

Similarly, a stochastic process  $X_t$  converges to a random variable  $X$  in  $L^2$  as  $t \rightarrow t_0$  if

$$\lim_{t \rightarrow t_0} \mathbb{E}[|X_t - X|^2] = 0.$$

**Problem 8.** Suppose that  $X_n \rightarrow X$  in  $L^2$ .

(a) Use the Cauchy–Schwarz inequality to show that

$$\mathbb{E}[|X_n - X|] \leq \sqrt{\mathbb{E}[|X_n - X|^2]}.$$

(b) Use part (a) to prove that

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

Thus, convergence in  $L^2$  implies convergence in expectation.

**Problem 9.** Let  $B_t$  be a standard Brownian motion and define, for  $t > 0$ ,

$$X_t = tB_{1/t}.$$

(a) Show that  $(X_{t_1}, \dots, X_{t_n})$  is a Gaussian random vector for every  $0 < t_1 < \dots < t_n$ .

(b) Compute  $\mathbb{E}[X_t]$  and  $\text{Cov}(X_s, X_t)$  and use these calculations to show that  $X_t$  has the same finite-dimensional distributions as standard Brownian motion on  $t > 0$ .

(c) Show that  $X_t \rightarrow 0$  in  $L^2$  as  $t \rightarrow 0^+$ .

(d) Use part (c) to explain how one may extend  $X_t$  to  $t = 0$  so that the resulting process has the same finite-dimensional distributions as standard Brownian motion on  $t \geq 0$ .

**Problem 5.** Let  $B_t$  be a standard Brownian motion and define

$$Z_t = B_t - tB_1, \quad 0 \leq t \leq 1.$$

For  $\Delta t > 0$ , define  $\Delta Z_t = Z_{t+\Delta t} - Z_t$ , whenever  $t + \Delta t \leq 1$ .

- (a) Compute the exact values of  $Z_0$  and  $Z_1$ .
- (b) Compute  $\mathbb{E}[Z_t]$  and  $\text{Var}(Z_t)$ .
- (c) Compute  $\text{Cov}(Z_s, Z_t)$  for  $0 \leq s \leq t \leq 1$ .
- (d) Compute  $\mathbb{E}[\Delta Z_t]$  and  $\text{Var}(\Delta Z_t)$ .
- (e) Are increments of  $Z_t$  over disjoint time intervals independent? Justify your answer by computing an appropriate covariance.

**Problem 6.** Let  $B_t$  be a standard Brownian motion and define

$$Y_t = e^{-t/2} B_{e^t}, \quad t \in \mathbb{R}.$$

- (a) Show that  $Y_t$  is a Gaussian process with mean zero.
- (b) Compute  $\text{Var}(Y_t)$ .
- (c) Compute  $\text{Cov}(Y_s, Y_t)$  and show that it depends only on  $|t - s|$ .
- (d) Explain why  $Y_t$  is a stationary Gaussian process.

### Homework #5

#1

Let  $B_t$  be standard Brownian motion. If  $0 \leq s \leq t$ , use the properties of Brownian motion to compute each of the following:

(a)  $\text{Cov}(B_s, B_t - B_s)$

(b) The joint distribution of  $(B_s, B_t - B_s)$ .

Solution:

(a)  $\text{Cov}(B_s, B_t - B_s) = 0$

(b)  $(B_s, B_t - B_s) \sim \frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{x^2}{2s}\right) \exp\left(-\frac{y^2}{2(t-s)}\right)$

#2

Let  $B_t$  be standard Brownian motion and let

$$X_t = \mu t + \sigma B_t$$

where  $\mu \in \mathbb{R}$  and  $\sigma > 0$ .

(a) Compute  $E[X_t]$ ,  $\text{Var}(X_t)$ , and  $\text{Cov}(X_s, X_t)$  for  $0 \leq s \leq t$ .

(b) For  $0 \leq s < t$ , determine the distribution of the increment

$$X_t - X_s$$

(c) For  $0 \leq t_1 < t_2 < t_3 < t_4$ , determine whether the increments

$$X_{t_2} - X_{t_1} \text{ and } X_{t_4} - X_{t_3}$$

(d) Write down the probability density of  $X_t$  for  $t > 0$ .

Solution:

(a)  $E[X_t] = E[\mu t + \sigma B_t] = E[\mu t] + \sigma E[B_t] = \mu t$

$$\text{Cov}(X_s, X_t) = E[(X_s - \mu s)(X_t - \mu t)] = E[\sigma B_s \sigma B_t] = \sigma^2 s$$

$$\text{Var}(X_t) = \text{Cov}(X_t, X_t) = \sigma^2 t$$

$$(b) \mathbb{E}[X_t - X_s] = \mathbb{E}[X_t] - \mathbb{E}[X_s] = \mu(t-s)$$

$$\text{Var}(X_t - X_s) = \mathbb{E}[(X_t - X_s - \mu(t-s))(X_t - X_s - \mu(t-s))] = \mathbb{E}[(B_t - B_s)(B_t - B_s)] = t-s.$$

$$\Rightarrow X_t - X_s \sim \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x - \mu(t-s))^2}{2(t-s)}\right)$$

$$(c) \text{Cov}(X_t - X_s - \mu(t-s), X_t - X_s - \mu(t-s)) = \text{Cov}(B_t - B_s, B_t - B_s) = 0.$$

Therefore,  $X_t - X_s$  and  $X_t - X_s - \mu(t-s)$  are independent.

$$(d) X_t \sim \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - \mu t)^2}{2t}\right)$$

#3

Let  $B_t$  be standard Brownian motion. Compute the following

$$(a) \mathbb{E}[B_t^4]$$

$$(b) \mathbb{E}[B_s B_t] \text{ if } 0 < s < t$$

$$(c) \mathbb{E}[B_s^3 B_t^2] \text{ if } 0 < s < t$$

$$(d) \mathbb{E}[B_s B_t^3] \text{ if } 0 < s < t$$

$$(e) \mathbb{E}[(B_t - B_s)^4] \text{ if } 0 < s < t$$

Solution:

$$(a) \mathbb{E}[B_t^4] = 3!! t^2 = 3t^2$$

$$(b) \mathbb{E}[B_s B_t] = \mathbb{E}[B_s (B_t - B_s + B_s)] = \mathbb{E}[B_s (B_t - B_s)] + \mathbb{E}[B_s^2] = \mathbb{E}[B_s^2] = s.$$

$$\begin{aligned} (c) \mathbb{E}[B_s^3 B_t^2] &= \mathbb{E}[B_s^3 (B_t - B_s + B_s)^2] = \mathbb{E}[B_s^3 ((B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2)] \\ &= \mathbb{E}[B_s^3 (B_t - B_s)^2] + 2\mathbb{E}[(B_t - B_s)B_s^4] + \mathbb{E}[B_s^5] \\ &= \mathbb{E}[B_s^3] \mathbb{E}[(B_t - B_s)^2] + 2\mathbb{E}[(B_t - B_s)] \mathbb{E}[B_s^4] + 3s^2 \\ &= 5(t-s) + 3s^2 \\ &= 5(t+2s) \end{aligned}$$

$$(d) E[B_s B_t^3] = E[B_s (B_t - B_s + B_s)^3]$$

$$= E[B_s ((B_t - B_s)^3 + 3(B_t - B_s)^2 B_s + 3(B_t - B_s) B_s^2 + B_s^3)]$$

$$= E[B_s (B_t - B_s)^3] + 3E[B_s^2 (B_t - B_s)^2] + 3E[(B_t - B_s) B_s^3] + E[B_s^4]$$

$$= E[B_s] E[(B_t - B_s)^3] + 3E[B_s^2] E[(B_t - B_s)^2] + 3E[B_t - B_s] E[B_s^3] + 3s^2$$

$$= 0 + 3s(t-s) + 3s^2$$

$$= 3st$$

$$(e) E[(B_t - B_s)^4] = 3(t-s)^2$$

#4

$$(a) P(B_1 > 1, B_2 > 1) = P(B_1 > 1, B_2 - B_1 > 1 - B_1)$$

$$= \frac{1}{2\pi} \int_1^\infty \int_{1-x}^\infty \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{y^2}{2}\right) dy dx$$

$$(b) P(B_1 > 0, B_2 > 1, B_3 > 2) = P(B_1 > 0, B_2 - B_1 > 1 - B_1, B_3 - B_2 > 2 - B_2)$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_{1-x}^\infty \int_{2-x-y}^\infty \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{y^2}{2}\right) \exp\left(-\frac{z^2}{2}\right) dz dy dx$$

#5

Let  $B_t$  be standard Brownian motion and define

$$Z_t = B_t - tB_1, \quad 0 \leq t \leq 1$$

(a) Compute the exact values of  $Z_0$  and  $Z_1$ .

(b) Compute  $E[Z_t]$  and  $\text{Var}(Z_t)$ .

(c) Compute  $\text{Cov}(Z_s, Z_t)$  for  $0 \leq s \leq t \leq 1$ .

(d) Compute  $E[\Delta Z_t]$  and  $\text{Var}(\Delta Z_t)$ .

(e) Are increments of  $Z_t$  over disjoint intervals independent?

Solution:

$$(a) Z_0 = B_0 - 0 \cdot B_1 = 0, Z_1 = B_1 - 1 \cdot B_1 = 0.$$

$$(b) E[Z_t] = E[B_t] - tE[B_1] = 0$$

$$\begin{aligned} \text{Var}(Z_t) &= E[(B_t - tB_1)^2] = E[B_t^2] - 2tE[B_t B_1] + t^2 E[B_1^2] \\ &= t - 2tE[(B_t - B_t + B_t)B_1] + t^2 \\ &= t - 2tE[(B_t - B_t)]E[B_1] - 2tE[B_t^2] + t^2 \\ &= t - 2t^2 + t^2 \\ &= t(1-t). \end{aligned}$$

$$\begin{aligned} (c) \text{Cov}(Z_s, Z_t) &= E[(B_t - tB_1)(B_s - sB_1)] \\ &= E[B_t B_s] - sE[B_t B_1] - tE[B_s B_1] + stE[B_1^2] \\ &= s - st - st + st \\ &= s(1-t) \end{aligned}$$

$$\begin{aligned} (d) E[\Delta Z_t] &= E[B_{t+\Delta t} - (t+\Delta t)B_1 - B_t + tB_1] \\ &= E[B_{t+\Delta t}] - (t+\Delta t)E[B_1] - E[B_t] + tE[B_1] \\ &= 0 \end{aligned}$$

$$\begin{aligned} E[\Delta Z_t^2] &= E[(B_{t+\Delta t} - B_t - \Delta t B_1)^2] \\ &= E[(B_{t+\Delta t} - B_t)^2 - 2\Delta t B_1 (B_{t+\Delta t} - B_t) + \Delta t^2 B_1^2] \\ &= \Delta t - 2\Delta t(t+\Delta t) + 2\Delta t t + \Delta t^2 \\ &= \Delta t - \Delta t^2 \\ &= \Delta t(1-\Delta t) \end{aligned}$$

(e) Let  $0 \leq t_0 < t_1 < t_2 < t_3$ . Therefore,

$$\begin{aligned} \text{Cov}(Z_{t_1} - Z_{t_0}, Z_{t_3} - Z_{t_2}) &= E[(Z_{t_1} - Z_{t_0})(Z_{t_3} - Z_{t_2})] \\ &= E[(B_{t_1} - B_{t_0} - (t_1 - t_0)B_1)(B_{t_3} - B_{t_2} - (t_3 - t_2)B_1)] \\ &= E[(B_{t_1} - B_{t_0})(B_{t_3} - B_{t_2})] - (t_3 - t_2)E[(B_{t_1} - B_{t_0})B_1] \\ &\quad - (t_1 - t_0)E[B_1(B_{t_3} - B_{t_2})] + (t_1 - t_0)(t_3 - t_2)E[B_1^2] \\ &= -(t_3 - t_2)(t_1 - t_0) - (t_1 - t_0)(t_3 - t_2) + (t_1 - t_0)(t_3 - t_2) \\ &= -(t_3 - t_1)(t_1 - t_0) \end{aligned}$$

#6

Let  $B_t$  be standard Brownian motion and define

$$Y_t = e^{-t/2} B_{e^t}$$

(a) Show that  $Y_t$  is a Gaussian process with mean 0.

(b) Compute  $\text{Var}(Y_t)$ .

(c) Compute  $\text{Cov}(Y_s, Y_t)$  and show that it depends only on  $|t-s|$ .

Solution:

(a)  $B_t$  is Gaussian hence  $Y_t = e^{-t/2} B_{e^t}$  is as well. We also have  $E[e^{-t/2} B_{e^t}] = e^{-t/2} E[B_{e^t}] = 0$ .

(b)  $\text{Var}(Y_t) = E[Y_t^2] = E[e^{-t} B_{e^t}^2] = e^{-t} e^t = 1$ .

(c)  $\text{Cov}(Y_s, Y_t) = E[Y_s Y_t]$

$$= e^{-s/2} e^{-t/2} E[B_{e^s} B_{e^t}]$$

$$= e^{-s/2} e^{-t/2} \min\{e^s, e^t\}$$

$$= \begin{cases} e^{-(t-s)}, & t > s \\ e^{-(s-t)}, & t \leq s \end{cases}$$

$$= e^{-|t-s|}$$

#7

Let  $X$  and  $Y$  be random variables satisfying  $E[X^2], E[Y^2] < \infty$ . Prove the Cauchy-Schwarz inequality:

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

proof:

$$\frac{d}{da} E[(X - aY)^2] = -2E[(X - aY)Y] = 0$$

$$\Rightarrow E[XY] = a E[Y^2]$$

$$\Rightarrow a = E[XY] / E[Y^2]$$

Since  $E[(X-aY)^2]$  is quadratic in  $a$  and hence minimized when  $a = \frac{E[XY]}{E[Y^2]}$   
 it follows that for all  $a \in \mathbb{R}$

$$E[(X-aY)^2] \geq E\left[\left(X - \frac{E[XY]}{E[Y^2]} Y\right)^2\right] \geq 0$$

$$\Rightarrow E[X^2] - 2 \frac{E[XY]E[XY]}{E[Y^2]} + \frac{E[XY]^2 E[Y^2]}{E[Y^2]^2} \geq 0$$

$$\Rightarrow E[X^2]E[Y^2] \geq E[XY]^2$$

$$\Rightarrow |E[XY]| \leq \sqrt{E[X^2]} \sqrt{E[Y^2]}$$

#8

Suppose  $X_n \rightarrow X$  in  $L^2$

(a) Use Cauchy-Schwarz to show that

$$E[|X_n - X|] \leq \sqrt{E[|X_n - X|^2]}$$

(b) Use part (a) to prove that

$$\lim_{n \rightarrow \infty} E[X_n] = E[X]$$

proof:

$$(a) E[|X_n - X|] = E[|X_n - X| \cdot 1] \leq (E[|X_n - X|^2])^{1/2} (E[1^2])^{1/2} = E[|X_n - X|^2]^{1/2}$$

$$(b) |E[X_n] - E[X]| = |E[X_n - X]| \leq E[|X_n - X|] \leq E[|X_n - X|^2]^{1/2}$$

Therefore, by the squeeze theorem,

$$\lim_{n \rightarrow \infty} E[X_n] = E[X]$$

#9

Let  $B_t$  be standard Brownian motion and define, for  $t > 0$ ,  
$$\tilde{X}_t = t B_{1/t}.$$

- (a) Show that  $(\tilde{X}_t, \dots, \tilde{X}_n)$  is a Gaussian vector.
- (b) Compute  $\mathbb{E}[\tilde{X}_t]$  and  $\text{Cov}(\tilde{X}_s, \tilde{X}_t)$  and use these calculations to show that  $\tilde{X}_t$  has the same distribution as standard Brownian motion on  $t > 0$ .
- (c) Show that  $\tilde{X}_t \rightarrow 0$  in  $L^2$  as  $t \rightarrow 0^+$ .
- (d) Use part (b) to extend  $\tilde{X}_t = 0$  to  $t = 0$  so that it is a Brownian motion on  $t \geq 0$ .

Solution:

(a) This is clear.

(b)  $\mathbb{E}[\tilde{X}_t] = t \mathbb{E}[B_{1/t}] = 0$

$\text{Cov}(\tilde{X}_s, \tilde{X}_t) = \mathbb{E}[\tilde{X}_s \tilde{X}_t] = st \mathbb{E}[B_{1/s} B_{1/t}] = st \min\{1/s, 1/t\} = \min\{s, t\}.$

(c)  $\text{Var}(\tilde{X}_t) = t$  and thus  $\tilde{X}_t \rightarrow 0^+$  in  $L^2$ .

(d) Define

$$\tilde{X}_t = \begin{cases} 0, & t = 0 \\ t B_{1/t}, & t > 0 \end{cases}$$