Homework #6, Due Date: 10/23/26
MTH 383/683: Fall 2026
Instructor: John Gemmer

1 Problems for all Students

Problem 1. Let (X, Y) be two random variables with joint density $f(x, y)$ on $\mathbb{R}^2$. Recall that the marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy.$$

Assume that f and f_X are continuous and that $f_X(x) > 0$ at the value of x under consideration.

(a) Let $a < b$ and $\varepsilon > 0$. Using the definition of conditional probability, show that

$$P(a < Y < b | x < X < x + \varepsilon) = \frac{\int_x^{x+\varepsilon} \int_a^b f(u, y)\, dy\, du}{\int_x^{x+\varepsilon} f_X(u)\, du}.$$

(b) By taking the limit as $\varepsilon \to 0^+$, show that

$$\lim_{\varepsilon \to 0^+} P(a < Y < b | x < X < x + \varepsilon) = \frac{\int_a^b f(x, y)\, dy}{f_X(x)}.$$

Hint: Use the Fundamental Theorem of Calculus.

(c) Using part (b), explain why the conditional density of Y given $X = x$ is

$$f_{Y|X=x}(y) = \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, z)\, dz}.$$

(d) Prove that $f_{Y|X=x}(y)$ is a probability density.

Problem 2. Let (X, Y) be two random variables with joint density $f(x, y)$ on $\mathbb{R}^2$. Recall that conditional expectation is the random variable $h(X) = \mathbb{E}[Y|X]$ defined by

$$h(x) = \frac{\int_{-\infty}^{\infty} y f(x, y)\, dy}{\int_{-\infty}^{\infty} f(x, y)\, dy},$$

for values of x where the denominator is positive. Assume throughout that all relevant expectations are well defined.

(a) Explain why this definition follows naturally from Problem #1.

(b) Prove that for every bounded measurable function g, $\mathbb{E}[g(X)\mathbb{E}[Y|X]] = \mathbb{E}[g(X)Y]$.

(c) Use part (b) to prove the law of total expectation, $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

(d) Prove that for constants $a, b \in \mathbb{R}$, $\mathbb{E}[aY + b|X] = a\mathbb{E}[Y|X] + b$.

(e) Prove that for every bounded measurable function g, $\mathbb{E}[g(X)Y|X] = g(X)\mathbb{E}[Y|X]$. In particular, show that $\mathbb{E}[g(X)|X] = g(X)$.

(f) Suppose that X and Y are independent. Prove that $\mathbb{E}[Y|X] = \mathbb{E}[Y]$.

(g) Suppose that $Y \geq 0$. Prove that $\mathbb{E}[Y|X] \geq 0$.

Problem 3. Let (X, Y) be jointly Gaussian random variables with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \qquad C = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix},$$

where $\sigma_X^2 > 0$. We would like to show that

$$\mathbb{E}[Y|X] = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X).$$

(a) Let $Z = Y - \mu_Y - \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X)$. Show that $\mathbb{E}[Z] = 0$ and $\mathrm{Cov}(X, Z) = 0$. Explain why X and Z are independent. Recall that uncorrelated jointly Gaussian random variables are independent.

(b) Let $h(X) = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X)$. Using the independence of X and Z, prove that for every bounded measurable function g,

$$\mathbb{E}[g(X)Y] = \mathbb{E}[g(X)h(X)].$$

Conclude that $\mathbb{E}[Y|X] = h(X)$.

Problem 4. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion. In each part, use the result of Problem #3 to compute the indicated conditional expectations.

(a) Let $Z_t = tB_t$ for $t > 0$. For $0 < s < t$, compute $\mathbb{E}[Z_t|Z_s]$ and $\mathbb{E}[Z_s|Z_t]$.

(b) Let $W_t = e^{-t}B_t$ for $t \geq 0$. For $0 < s < t$, compute $\mathbb{E}[W_t|W_s]$ and $\mathbb{E}[W_s|W_t]$.

(c) Let $U_t = B_t - tB_1$ for $0 \leq t \leq 1$ be a Brownian bridge. For $0 < s < t < 1$, compute $\mathbb{E}[U_t|U_s]$ and $\mathbb{E}[U_s|U_t]$.

(d) Let $V_t = B_{t^2}$ for $t \geq 0$. For $0 < s < t$, compute $\mathbb{E}[V_s|V_t]$ and $\mathbb{E}[V_t|V_s]$.

(e) Let $X_t = \mu t + \sigma B_t$, where $\mu \in \mathbb{R}$ and $\sigma > 0$. For $0 < s < t$, compute $\mathbb{E}[X_t|X_s]$ and $\mathbb{E}[X_s|X_t]$.

2 Problems for Graduate Students

Definition: Quadratic Variation

Let X_t be a stochastic process on $[0, T]$. Given a partition

$$\mathcal{P}_n = \{0 = t_0 < t_1 < \cdots < t_n = T\},$$

define the quadratic variation sum

$$Q_n = \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2.$$

The quadratic variation of X_t along a sequence of partitions whose mesh tends to zero is the limit of Q_n, provided the limit exists in the specified sense.
Note: The type of convergence must be specified. In the previous homework, we defined convergence in L^2.

Definition: Convergence in Probability

A sequence of random variables $\{X_n\}$ converges to a random variable X in probability if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0.$$

Problem 5. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion on $[0, T]$, where $T > 0$. For each $n \geq 1$, consider the uniform partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \qquad t_i = \frac{iT}{n}.$$

Define

$$Q_n = \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2.$$

(a) Using the properties of Brownian motion, prove that

$$\mathbb{E}[Q_n] = T.$$

(b) Recall that if $X \sim N(0, \sigma^2)$, then $\mathbb{E}[X^4] = 3\sigma^4$. Use this fact to prove that

$$\mathrm{Var}(Q_n) = \frac{2T^2}{n}.$$

(c) Prove that $Q_n \longrightarrow T$ in L^2.

(d) Use Chebyshev's inequality to prove that $Q_n \longrightarrow T$ in probability.

Definition: Total Variation

The total variation of a function $f : [0, T] \to \mathbb{R}$ is defined by

$$V_T(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $\mathcal{P}$ of $[0, T]$. A function f has finite total variation on $[0, T]$ if $V_T(f) < \infty$.

Definition: Arc Length

Let $f : [0, T] \to \mathbb{R}$ be continuous. The arc length of the graph of f is defined by

$$L_T(f) = \sup_{\mathcal{P}} \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2},$$

where the supremum is taken over all finite partitions $\mathcal{P}$ of $[0, T]$. If f is continuously differentiable, then

$$L_T(f) = \int_0^T \sqrt{1 + (f'(t))^2}\, dt.$$

Problem 6. In this problem, we will investigate the total variation and arc length of Brownian motion and explore how these properties relate to differentiability.

(a) Let $f : [0, T] \to \mathbb{R}$ be continuously differentiable. Use the Fundamental Theorem of Calculus to prove that

$$V_T(f) \leq \int_0^T |f'(t)|\, dt < \infty.$$

Explain why every continuously differentiable function on $[0, T]$ has finite total variation.

(b) Let $f : [0, T] \to \mathbb{R}$ be continuous. Prove that

$$V_T(f) \leq L_T(f) \leq T + V_T(f).$$

Conclude that a continuous function has finite arc length if and only if it has finite total variation.

(c) Suppose that $f : [0, T] \to \mathbb{R}$ is continuous and has finite total variation. Prove that for any sequence of partitions whose mesh tends to zero,

$$\lim_{n\to\infty} \sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2 = 0.$$

Hint: First show that

$$\sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2 \leq \max_{1\leq i\leq n} |f(t_i) - f(t_{i-1})| V_T(f).$$

Then use the uniform continuity of f on $[0, T]$.

(d) Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion. Using part (c) and the results of Problem #5, prove that for every $T > 0$,

$$P(V_T(B) = \infty) = 1.$$

Hint: Suppose that $V_T(B) < \infty$ on an event A of positive probability. By part (c), the quadratic variation sums Q_n from Problem #5 converge to zero on A. Show that for $\varepsilon = T/2$,

$$\liminf_{n\to\infty} P(|Q_n - T| > \varepsilon) \geq P(A).$$

Explain why this contradicts convergence in probability from Problem #5.

(e) Prove that the graph of Brownian motion on $[0, T]$ has infinite arc length with probability one, i.e.,

$$P(L_T(B) = \infty) = 1.$$

(f) Explain why the results above imply that Brownian motion cannot have continuously differentiable sample paths on $[0, T]$. Does this argument establish that Brownian motion is nowhere differentiable? Explain.

DATE SUBJECT

#1

Assume that f and f_X are continuous and that $f_X(x)>0$ at the value of x under consideration.

(a) Let $a<b$ and $\varepsilon>0$. Show that

$$P(a<Y<b \mid x<X<x+\varepsilon)=\frac{\int_x^{x+\varepsilon}\int_a^b f(u,y)\,dy\,du}{\int_x^{x+\varepsilon} f_X(u)\,du}$$

(b) Show that

$$\lim_{\varepsilon\to 0^+} P(a<Y<b \mid x<X<x+\varepsilon)=\frac{\int_a^b f(u,y)\,dy}{f_X(x)}$$

(c) Explain why the conditional density of Y given $X=x$ is

$$f_{Y|X=x}=\frac{f(x,y)}{\int_{-\infty}^{\infty} f(x,z)\,dz}$$

(d) Prove that $f_{Y|X=x}(y)$ is a probability density.

Solution:

(a) $$P(a<Y<b \mid x<X<x+\varepsilon)=\frac{P(a<Y<b,\ x<X<x+\varepsilon)}{P(x<X<x+\varepsilon)}$$
$$=\frac{\int_x^{x+\varepsilon}\int_a^b f(u,y)\,du\,dy}{\int_x^{x+\varepsilon} f_X(u)\,du}.$$

(b) Applying L'Hospital's rule we have that

$$\lim_{\varepsilon\to 0^+} P(a<Y<b \mid x<X<x+\varepsilon)=\lim_{\varepsilon\to 0^+}\frac{\int_x^{x+\varepsilon}\int_a^b f(u,y)\,du\,dy}{\int_x^{x+\varepsilon} f_X(u)\,du}$$
$$=\lim_{\varepsilon\to 0^+}\frac{\frac{d}{d\varepsilon}\int_x^{x+\varepsilon}\int_a^b f(u,y)\,du\,dy}{\frac{d}{d\varepsilon}\int_x^{x+\varepsilon} f_X(u)\,du}$$
$$=\lim_{\varepsilon\to 0^+}\frac{\int_a^b f(x+\varepsilon,y)\,dy}{f_X(x+\varepsilon)}$$
$$=\frac{\int_a^b f(x,y)\,dy}{f_X(x)}$$

(c) If we suppose Y given $X=x$ has a density $g(x)$ then

$$P(a<Y<b \mid X=x)=\int_a^b g(y)\,dy=\frac{\int_a^b f(u,y)\,dy}{f_X(x)} \Rightarrow g(y)=\frac{\int_a^b f(u,y)\,dy}{f_X(x)}.$$

(d) $\int_{-\infty}^{\infty} f_{Y|X=x}\,dy = \int_{-\infty}^{\infty} \frac{f(x,y)}{f_X(x)}\,dy = \frac{\int_{-\infty}^{\infty} f(x,y)\,dy}{f_X(x)} = \frac{f_X(x)}{f_X(x)} = 1$ ■

#2

Let (X,Y) be two random variables with joint density $f(x,y)$ on $\mathbb{R}^2$ and recall that conditional expection is the random variable $h(X)=\mathbb{E}[Y|X]$ defined by

$$h(x)=\frac{\int_{-\infty}^{\infty} y\,f(x,y)\,dy}{\int_{-\infty}^{\infty} f(x,y)\,dy}$$

(a) Explain why this definition follows from problem #1.
(b) Prove that for every bounded measurable function g, $\mathbb{E}[g(X)\mathbb{E}[Y|X]]=\mathbb{E}[g(X)Y]$
(c) Prove that $\mathbb{E}[\mathbb{E}[Y|X]]=\mathbb{E}[Y]$.
(d) Prove that $\mathbb{E}[aY+b|X]=a\,\mathbb{E}[Y|X]+b$
(e) Prove that $\mathbb{E}[g(X)Y|X]=g(X)\,\mathbb{E}[Y|X]$. In particular show that $\mathbb{E}[g(X)|X]=g(X)$.
(f) Suppose X and Y are independent. Prove that $\mathbb{E}[Y|X]=\mathbb{E}[Y]$.
(g) If $Y\geq 0$ prove that $\mathbb{E}[Y|X]\geq 0$.

Solution:

(a) From problem 1, we expect that

$$\mathbb{E}[Y|X]=\int_{-\infty}^{\infty} f_{Y|X=x}(y)\,dy=\int_{-\infty}^{\infty} \frac{y\,f(x,y)}{f_X(x)}\,dy=\frac{\int_{-\infty}^{\infty} y\,f(x,y)\,dy}{\int_{-\infty}^{\infty} f(x,y)\,dy}$$

(b) $\mathbb{E}[g(X)Y]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,y\,f(x,y)\,dy\,dx$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,y\,\frac{f_X(x)}{f_X(x)}\,f(x,y)\,dy\,dx$$

$$=\int_{-\infty}^{\infty} g(x)\,f_X(x)\left(\int_{-\infty}^{\infty}\frac{y\,f(x,y)}{f_X(x)}\,dy\right)dx$$

$$=\int_{-\infty}^{\infty} g(x)\,\mathbb{E}[Y|X]\,f_X(x)\,dx$$

$$=\mathbb{E}[g(X)\,\mathbb{E}[Y|X]].$$

(c) Setting $g=1$, we obtain $\mathbb{E}[\mathbb{E}[Y|X]]=\mathbb{E}[Y]$

(d) $\mathbb{E}[aY+b|X]=\dfrac{\int_{-\infty}^{\infty}(ay+b)f(x,y)dy}{f_X(x)}=\dfrac{a\int_{-\infty}^{\infty}yf(x,y)dy+b\int_{-\infty}^{\infty}f(x,y)}{f_X(x)}=a\mathbb{E}[Y|X]+b.$

(e) $\mathbb{E}[g(X)Y|X]=\dfrac{\int_{-\infty}^{\infty}g(x)yf(x,y)dy}{f_X(x)}=\dfrac{g(x)\int_{-\infty}^{\infty}yf(x,y)dy}{f_X(x)}=g(x)\mathbb{E}[Y|X]$

(f) If X and Y are independent then

$$\mathbb{E}[Y|X]=\frac{\int_{-\infty}^{\infty}yf(x,y)dy}{f_X(x)}=\frac{\int_{-\infty}^{\infty}yf_X(x)f_Y(y)dy}{f_X(x)}=\int_{-\infty}^{\infty}yf_Y(y)dy=\mathbb{E}[Y].$$

(g). If $Y\geq 0$ then $f(x,y)=0$ for $y<0$ and thus

$$\mathbb{E}[Y|X]=\frac{\int_{-\infty}^{\infty}yf(x,y)dy}{f_X(x)}=\frac{\int_{0}^{\infty}yf(x,y)dy}{f_X(x)}\geq 0.$$

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#3

For (X,Y) jointly Gaussian, prove that

$$\mathbb{E}[Y|X]=Y-\nu_Y-\frac{\sigma_{XY}}{\sigma_X^2}(X-\nu_X)$$

proof:

(a) First $\mathbb{E}[Y-\nu_Y-\sigma_{XY}/\sigma_X(X-\nu_X)]=\mathbb{E}[Y-\nu_Y]-\sigma_{XY}/\sigma_X^2\,\mathbb{E}[X-\nu_X]=0.$

Therefore, letting $Z=Y-\nu_Y-\sigma_{XY}/\sigma_X^2(X-\nu_X)$, we have that

$$\begin{aligned}\mathrm{Cov}(Z,X)&=\mathbb{E}[Z(X-\nu_X)]\\&=\mathbb{E}[ZX]-\nu_X\mathbb{E}[Z]\\&=\mathbb{E}[XY]-\nu_Y\mathbb{E}[X]-\sigma_{XY}/\sigma_X^2\,\mathbb{E}[X^2]+\sigma_{XY}/\sigma_X^2\,\nu_Y\mathbb{E}[X]\\&=\sigma_{XY}+\nu_X\nu_Y-\nu_X\nu_X-\sigma_{XY}/\sigma_X^2(\sigma_X^2+\nu_X^2)+\sigma_{XY}/\sigma_X^2\,\nu_X^2\\&=0\end{aligned}$$

Consequently, since Z and X are Gaussian they are independent.

(b). Since X and Z are independent, we have that for every measurable function g that

$$\mathbb{E}[g(X)Z]=0$$
$$\Rightarrow\mathbb{E}[g(X)(Y-\nu_Y-\sigma_{XY}/\sigma_X^2(X-\nu_X))]=0$$
$$\Rightarrow\mathbb{E}[g(X)Y]=\mathbb{E}[g(X)(\nu_Y-\sigma_{XY}/\sigma_X^2(X-\nu_X))]$$

By definition, $\mathbb{E}[Y|X]=\nu_Y-\sigma_{XY}/\sigma_X^2(X-\nu_X)$

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#4

Let B_t be Brownian motion. Compete the indicated conditional expectations.

(a) $Z_t = tB_t$, $\mathbb{E}[Z_t|Z_s]$, $\mathbb{E}[Z_s|Z_t]$

(b) $W_t = e^{-t}B_t$, $\mathbb{E}[W_t|W_s]$, $\mathbb{E}[W_s|W_t]$

(c) $U_t = B_t - B_1$, $\mathbb{E}[U_t|U_s]$, $\mathbb{E}[U_s|U_t]$

(d) $V_t = B_t^2$, $\mathbb{E}[V_t|V_s]$, $\mathbb{E}[V_s|V_t]$

(e) $X_t = \nu t + \sigma B_t$, $\mathbb{E}[X_t|X_s]$, $\mathbb{E}[X_s|X_t]$

Solution:

(a) $\mathbb{E}[Z_t|Z_s] = \frac{\mathbb{E}[Z_tZ_s]}{\mathbb{E}[Z_s^2]} Z_s = \frac{\mathbb{E}[ts\,B_tB_s]}{\mathbb{E}[s^2B_s]} Z_s = \frac{ts^2}{s^3} Z_s = \frac{t}{s} Z_s = tB_s$

$\mathbb{E}[Z_s|Z_t] = \frac{\mathbb{E}[ts\,B_tB_s]}{\mathbb{E}[t^2B_t^2]} Z_t = \frac{ts^2}{t^3} Z_t = \frac{s^2}{t^2} Z_t = \frac{s^2}{t} B_t$

(b) $\mathbb{E}[W_t|W_s] = \frac{\mathbb{E}[e^{-t}e^{-s}B_sB_t]}{\mathbb{E}[e^{-2s}B_s^2]} W_s = \frac{e^{-t}e^{-s}s}{e^{-2s}\cdot s} W_s = e^{-(t-s)}e^{-s}B_s = e^{-t}B_s$

$\mathbb{E}[W_s|W_t] = \frac{\mathbb{E}[e^{-t}e^{-s}B_sB_t]}{\mathbb{E}[e^{-2t}B_t^2]} W_t = \frac{e^{-t}e^{-s}\cdot s}{e^{-2t}t} W_t = \frac{s}{t} e^{-s}B_t.$

(c) $\mathbb{E}[U_t|U_s] = \frac{\mathbb{E}[(B_t-B_1)(B_s-B_1)]}{\mathbb{E}[(B_s-B_1)^2]} U_s = \frac{(s-s-t+1)}{1-s} U_s = \frac{1-t}{1-s}(B_s-B_1)$

$\mathbb{E}[U_s|U_t] = \frac{\mathbb{E}[(B_t-B_1)(B_s-B_1)]}{\mathbb{E}[(B_t-B_1)^2]} U_t = \frac{1-t}{1-t}(B_t-1) = B_t-1$

(d) $\mathbb{E}[V_t|V_s] = \frac{\mathbb{E}[B_t^2B_s^2]}{\mathbb{E}[B_s^2]} V_s = \frac{s^2}{s^2} B_s^2 = B_s^2$

$\mathbb{E}[V_s|V_t] = \frac{\mathbb{E}[B_t^2B_s^2]}{\mathbb{E}[B_t^2]} V_t = \frac{s^2}{t^2} B_t^2$

(e) $\mathbb{E}[X_t|X_s] = \nu t + \frac{\mathbb{E}[\sigma B_t\sigma B_s]}{\mathbb{E}[\sigma^2B_s^2]}(\nu s+\sigma B_s-\nu s) = \nu t + \frac{\sigma^2 s}{\sigma^2 s}\cdot\sigma B_s = \nu t+\sigma B_s.$

$\mathbb{E}[X_s|X_t] = \nu s + \frac{\mathbb{E}[\sigma B_t\sigma B_s]}{\mathbb{E}[\sigma^2B_t^2]}(\nu t+\sigma B_t-\nu t) = \nu s + \frac{s}{t}\sigma B_t$

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#5

Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion on $[0,T]$, wher $T>0$. For each $n\geq 1$ consider the uniform partition

$$0=t_0<t_1<\dots<t_n=T,\ t_i=iT/n$$

Define

$$Q_n=\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2$$

(a) Prove that $\mathbb{E}[Q_n]=T$

(b) Prove that $Var(Q_n)=2T^2/n$.

(c) Prove that $Q_n\to T$ in L^2.

(d) Prove that $Q_n\to T$ in probability.

Solution:

(a) $\mathbb{E}[Q_n]=\mathbb{E}[\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2]=\sum_{i=1}^{n}\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2]=\sum_{i=1}^{n}(t_i-t_{i-1})=\sum_{i=1}^{n}T/n=T.$

(b) $Var(Q_n)=\mathbb{E}[Q_n^2]-T^2$

$$=\mathbb{E}[(\sum_{i=1}^{n}(B_{t_i}-B_{t_{i-1}})^2)^2]-T^2$$

$$=\mathbb{E}[\sum_{i=1}^{n}\sum_{j=1}^{n}(B_{t_i}-B_{t_{i-1}})^2(B_{t_j}-B_{t_{j-1}})^2]-T^2$$

$$=\sum_{i=1}^{n}\sum_{j\neq i}\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2]\mathbb{E}[(B_{t_j}-B_{t_{j-1}})^2]+\sum_{i=1}^{n}\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^4]-T^2$$

$$=\sum_{i=1}^{n}\sum_{j\neq i}T^2/n^2+\sum_{i=1}^{n}3T^2/n^2-T^2$$

$$=\frac{T^2n(n-1)}{n^2}+\frac{3T^2}{n}-T^2$$

$$=\frac{2T^2}{n}.$$

(c) By construction, $\lim_{n\to\infty}\mathbb{E}[(Q_n-T)^2]=\lim_{n\to\infty}Var(Q_n)=\lim_{n\to\infty}\frac{2T^2}{n}=0.$

(d) By Chebyshev's inequality

$$P(|Q_n-T|>\varepsilon)\leq\frac{1}{\varepsilon}\mathbb{E}[(Q_n-T)^2]=\frac{2T^2}{n\varepsilon}.$$

Therefore, $\lim_{n\to\infty}P(|Q_n-T|>\varepsilon)=0.$

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#6

In this problem, we investigate differentiability of Brownian motion.

(a) Let $f:[0,T]\to\mathbb{R}$ be continuously differentiable. Prove that

$$V_T(f)\le\int_0^T|f'(t)|dt<\infty.$$

(b) Let $f:[0,T]\to\mathbb{R}$ be continuous. Prove that

$$V_T(f)\le L_T(f)\le T+V_T(f)$$

(c) Suppose that $f:[0,T]\to\mathbb{R}$ is continuous has finite total variation. Prove that for any sequence of partitions

$$\lim_{n\to\infty}\sum_{i=1}^{n}(f(t_i)-f(t_{i-1}))^2=0.$$

(d) Let $\{B_t\}$ be standard Brownian motion. Prove that for every $T>0$,

$$P(V_T(B)=\infty)=1.$$

(e) Prove that the graph of Brownian motion on $[0,T]$ has infinite arc length with probability one,

$$P(L_T(B)=\infty)=1.$$

(f) Explain why the results above imply Brownian motion cannot have continuously differentiable sample paths on $[0,T]$. Does this argument establish that Brownian motion is nowhere differentiable.

Solution:

(a) By the mean value theorem there exists $c_i\in[t_{i-1},t_i]$ such that

$$|f(t_i)-f(t_{i-1})|=|f'(c_i)|\cdot(t_i-t_{i-1})$$

Consequently,

$$\begin{aligned}V_T(f)&=\sup_p\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|\\&=\sup_p\sum_{i=1}^{n}\left|\int_{t_{i-1}}^{t_i}f'(t)dt\right|\\&\le\sup_p\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|f'(t)|dt\\&=\sup_p\int_0^T|f'(t)|dt\\&=\int_0^T|f'(t)|dt\end{aligned}$$

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(b) By construction,

$$L_T(f)=\sup_p \sum_{i=1}^{n} \sqrt{(t_i-t_{i-1})^2+(f(t_i)-f(t_{i-1})^2} \leq \sup_p \left(\sum_{i=1}^{n} (t_i-t_{i-1}) + \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})| \right)$$

$$\Rightarrow L_T(f) \leq T + V_T(f).$$

We also have that

$$L_T(f)=\sup_p \sum_{i=1}^{n} \sqrt{(t_i-t_{i-1})^2+(f(t_i)-f(t_{i-1}))^2} \geq \sup_p \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})|$$

$$\Rightarrow L_T(f) \geq V_T(f).$$

Therefore,

$$V_T(f) \leq L_T(f) \leq T + V_T(f).$$

(c) $$\sum_{i=1}^{n} (f(t_i)-f(t_{i-1}))^2 \leq \sum_{i=1}^{n} \max_j |f(t_j)-f(t_{j-1})|\,|f(t_i)-f(t_{i-1})|$$

$$= \max_j |f(t_j)-f(t_{j-1})| \sum_{i=1}^{n} |f(t_i)-f(t_{i-1})|$$

$$\leq \max_j |f(t_j)-f(t_{j-1})| \cdot V_T(f).$$

Therefore, by uniform continuity we have that $\lim_{n\to\infty} \max_{1\leq j\leq n} |f(t_j)-f(t_{j-1})|=0$. and thus

$$\lim_{n\to\infty} \sum_{i=1}^{n} (f(t_i)-f(t_{i-1}))^2 = 0.$$

(d) Suppose $V_T(B_t)<\infty$ on an event A satisfying $P(A)>0$. Therefore, for $\omega \in A$ it follows that $Q_n(B_t(\omega)) \to 0$ and thus there exists $N(\omega) \in \mathbb{N}$ such that

$$|Q_n(\omega)-T| \geq T/2$$

Consequently, $P(|Q_n(\omega)-T|>T/2) \geq P(A)$. However, we know

$$0 = \lim_{n\to\infty} P(|Q_n(\omega)-T|>T/2) \geq P(A)$$

Therefore, $P(A)=0$.

*Note: I really need Fatou's lemma.

(e) Since $P(V_T(B_t)=\infty)=1$ and $L_T(B_t) \geq V_T(B_t)$ it follows that $P(L_T(B_T)=\infty)=1$.

(f). If B_t was continuously differentiable it would have finite arclength. However, we showed it has infinite arclength on <u>every</u> interval implying it can be differentiable <u>nowhere</u>.