

1 Problems for all Students

Problem 1. Let (X, Y) be two random variables with joint density $f(x, y)$ on $\mathbb{R}^2$. Recall that the marginal density of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Assume that f and f_X are continuous and that $f_X(x) > 0$ at the value of x under consideration.

(a) Let $a < b$ and $\varepsilon > 0$. Using the definition of conditional probability, show that

$$P(a < Y < b | x < X < x + \varepsilon) = \frac{\int_x^{x+\varepsilon} \int_a^b f(u, y) dy du}{\int_x^{x+\varepsilon} f_X(u) du}.$$

(b) By taking the limit as $\varepsilon \rightarrow 0^+$, show that

$$\lim_{\varepsilon \rightarrow 0^+} P(a < Y < b | x < X < x + \varepsilon) = \frac{\int_a^b f(x, y) dy}{f_X(x)}.$$

Hint: Use the Fundamental Theorem of Calculus.

(c) Using part (b), explain why the conditional density of Y given $X = x$ is

$$f_{Y|X=x}(y) = \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, z) dz}.$$

(d) Prove that $f_{Y|X=x}(y)$ is a probability density.

Problem 2. Let (X, Y) be two random variables with joint density $f(x, y)$ on $\mathbb{R}^2$. Recall that conditional expectation is the random variable $h(X) = \mathbb{E}[Y|X]$ defined by

$$h(x) = \frac{\int_{-\infty}^{\infty} y f(x, y) dy}{\int_{-\infty}^{\infty} f(x, y) dy},$$

for values of x where the denominator is positive. Assume throughout that all relevant expectations are well defined.

(a) Explain why this definition follows naturally from Problem #1.

(b) Prove that for every bounded measurable function g , $\mathbb{E}[g(X)\mathbb{E}[Y|X]] = \mathbb{E}[g(X)Y]$.

(c) Use part (b) to prove the law of total expectation, $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$.

(d) Prove that for constants $a, b \in \mathbb{R}$, $\mathbb{E}[aY + b|X] = a\mathbb{E}[Y|X] + b$.

(e) Prove that for every bounded measurable function g , $\mathbb{E}[g(X)Y|X] = g(X)\mathbb{E}[Y|X]$. In particular, show that $\mathbb{E}[g(X)|X] = g(X)$.

(f) Suppose that X and Y are independent. Prove that $\mathbb{E}[Y|X] = \mathbb{E}[Y]$.

(g) Suppose that $Y \geq 0$. Prove that $\mathbb{E}[Y|X] \geq 0$.

Problem 3. Let (X, Y) be jointly Gaussian random variables with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix},$$

where $\sigma_X^2 > 0$. We would like to show that

$$\mathbb{E}[Y|X] = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X).$$

- (a) Let $Z = Y - \mu_Y - \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X)$. Show that $\mathbb{E}[Z] = 0$ and $\text{Cov}(X, Z) = 0$. Explain why X and Z are independent. Recall that uncorrelated jointly Gaussian random variables are independent.
- (b) Let $h(X) = \mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(X - \mu_X)$. Using the independence of X and Z , prove that for every bounded measurable function g ,

$$\mathbb{E}[g(X)Y] = \mathbb{E}[g(X)h(X)].$$

Conclude that $\mathbb{E}[Y|X] = h(X)$.

Problem 4. Let $\{B_t\}_{t \geq 0}$ be standard Brownian motion. In each part, use the result of Problem #3 to compute the indicated conditional expectations.

- (a) Let $Z_t = tB_t$ for $t > 0$. For $0 < s < t$, compute $\mathbb{E}[Z_t|Z_s]$ and $\mathbb{E}[Z_s|Z_t]$.
- (b) Let $W_t = e^{-t}B_t$ for $t \geq 0$. For $0 < s < t$, compute $\mathbb{E}[W_t|W_s]$ and $\mathbb{E}[W_s|W_t]$.
- (c) Let $U_t = B_t - tB_1$ for $0 \leq t \leq 1$ be a Brownian bridge. For $0 < s < t < 1$, compute $\mathbb{E}[U_t|U_s]$ and $\mathbb{E}[U_s|U_t]$.
- (d) Let $V_t = B_{t^2}$ for $t \geq 0$. For $0 < s < t$, compute $\mathbb{E}[V_s|V_t]$ and $\mathbb{E}[V_t|V_s]$.
- (e) Let $X_t = \mu t + \sigma B_t$, where $\mu \in \mathbb{R}$ and $\sigma > 0$. For $0 < s < t$, compute $\mathbb{E}[X_t|X_s]$ and $\mathbb{E}[X_s|X_t]$.

2 Problems for Graduate Students

Definition: Quadratic Variation

Let X_t be a stochastic process on $[0, T]$. Given a partition

$$\mathcal{P}_n = \{0 = t_0 < t_1 < \cdots < t_n = T\},$$

define the quadratic variation sum

$$Q_n = \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2.$$

The quadratic variation of X_t along a sequence of partitions whose mesh tends to zero is the limit of Q_n , provided the limit exists in the specified sense.

Note: The type of convergence must be specified. In the previous homework, we defined convergence in L^2 .

Definition: Convergence in Probability

A sequence of random variables $\{X_n\}$ converges to a random variable X in probability if, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \varepsilon) = 0.$$

Problem 5. Let $\{B_t\}_{t \geq 0}$ be standard Brownian motion on $[0, T]$, where $T > 0$. For each $n \geq 1$, consider the uniform partition

$$0 = t_0 < t_1 < \cdots < t_n = T, \quad t_i = \frac{iT}{n}.$$

Define

$$Q_n = \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2.$$

(a) Using the properties of Brownian motion, prove that

$$\mathbb{E}[Q_n] = T.$$

(b) Recall that if $X \sim N(0, \sigma^2)$, then $\mathbb{E}[X^4] = 3\sigma^4$. Use this fact to prove that

$$\text{Var}(Q_n) = \frac{2T^2}{n}.$$

(c) Prove that $Q_n \rightarrow T$ in L^2 .

(d) Use Chebyshev's inequality to prove that $Q_n \rightarrow T$ in probability.

Definition: Total Variation

The total variation of a function $f : [0, T] \rightarrow \mathbb{R}$ is defined by

$$V_T(f) = \sup_{\mathcal{P}} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $\mathcal{P}$ of $[0, T]$. A function f has finite total variation on $[0, T]$ if $V_T(f) < \infty$.

Definition: Arc Length

Let $f : [0, T] \rightarrow \mathbb{R}$ be continuous. The arc length of the graph of f is defined by

$$L_T(f) = \sup_{\mathcal{P}} \sum_{i=1}^n \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2},$$

where the supremum is taken over all finite partitions $\mathcal{P}$ of $[0, T]$. If f is continuously differentiable, then

$$L_T(f) = \int_0^T \sqrt{1 + (f'(t))^2} dt.$$

Problem 6. In this problem, we will investigate the total variation and arc length of Brownian motion and explore how these properties relate to differentiability.

- (a) Let $f : [0, T] \rightarrow \mathbb{R}$ be continuously differentiable. Use the Fundamental Theorem of Calculus to prove that

$$V_T(f) \leq \int_0^T |f'(t)| dt < \infty.$$

Explain why every continuously differentiable function on $[0, T]$ has finite total variation.

- (b) Let $f : [0, T] \rightarrow \mathbb{R}$ be continuous. Prove that

$$V_T(f) \leq L_T(f) \leq T + V_T(f).$$

Conclude that a continuous function has finite arc length if and only if it has finite total variation.

- (c) Suppose that $f : [0, T] \rightarrow \mathbb{R}$ is continuous and has finite total variation. Prove that for any sequence of partitions whose mesh tends to zero,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n (f(t_i) - f(t_{i-1}))^2 = 0.$$

Hint: First show that

$$\sum_{i=1}^n (f(t_i) - f(t_{i-1}))^2 \leq \max_{1 \leq i \leq n} |f(t_i) - f(t_{i-1})| V_T(f).$$

Then use the uniform continuity of f on $[0, T]$.

- (d) Let $\{B_t\}_{t \geq 0}$ be standard Brownian motion. Using part (c) and the results of Problem #5, prove that for every $T > 0$,

$$P(V_T(B) = \infty) = 1.$$

Hint: Suppose that $V_T(B) < \infty$ on an event A of positive probability. By part (c), the quadratic variation sums Q_n from Problem #5 converge to zero on A . Show that for $\varepsilon = T/2$,

$$\liminf_{n \rightarrow \infty} P(|Q_n - T| > \varepsilon) \geq P(A).$$

Explain why this contradicts convergence in probability from Problem #5.

- (e) Prove that the graph of Brownian motion on $[0, T]$ has infinite arc length with probability one, i.e.,

$$P(L_T(B) = \infty) = 1.$$

- (f) Explain why the results above imply that Brownian motion cannot have continuously differentiable sample paths on $[0, T]$. Does this argument establish that Brownian motion is nowhere differentiable? Explain.