

Lecture #7: Manipulations with Brownian Motion

Theorem/Definition: A process B_t defined on $(\Omega, \mathcal{F}, P)$ has the distribution of a Brownian motion on $[0, \infty)$ if and only if the following hold:

(i) $B_0 = 0$

(ii) For any $s < t$, the increment $B_t - B_s$ is Gaussian with mean 0 and variance $t - s$.

(iii) For any $n \in \mathbb{N}$, $0 < t_1 < t_2 < \dots < t_n < \infty$ the increments $(B_{t_1} - B_0), (B_{t_2} - B_{t_1}), \dots, (B_{t_n} - B_{t_{n-1}})$ are independent.

(iv) The path $t \rightarrow B_t(\omega)$ is continuous with probability 1.

(There exists $A \in \mathcal{F}$ such that $P(A) = 1$ and for all $\omega \in A$, $B_t(\omega)$ is continuous)

Examples

1. If $t_2 > t_1$, what is $P(B_{t_2} > B_{t_1})$?

$$\Rightarrow P(B_{t_2} - B_{t_1} > 0) = \frac{1}{\sqrt{2\pi(t_2 - t_1)^2}} \int_0^{\infty} e^{-b^2 / 2(t_2 - t_1)^2} db = \frac{1}{2}$$

2. What is $P(B_1 > 2, B_2 > 3)$?

• Approach:

$$\text{Cov}(B_1, B_1) = 1$$

$$\text{Cov}(B_1, B_2) = 1$$

$$\text{Cov}(B_2, B_2) = 2$$

$$\Rightarrow C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } C^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

The joint density is therefore

$$f(b_1, b_2) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left(-\frac{b^T C^{-1} b}{2}\right) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(2b_1^2 - 2b_1b_2 + b_2^2)\right)$$

$$\Rightarrow P(B_1 > 2, B_2 > 3) = \frac{1}{2\pi} \int_2^\infty \int_3^\infty \exp\left(-\frac{1}{2}(b_1^2 - 2b_1b_2 + b_2^2)\right) db_2 db_1$$

3. What is $P(B_1 > 2, B_2 - B_1 > 1)$?

$$P(B_1 > 2, B_2 - B_1 > 1) = P(B_1 - B_0 > 2, B_2 - B_1 > 1) = P(B_1 - B_0 > 2)P(B_2 - B_1 > 1)$$

$$\Rightarrow P(B_1 > 2, B_2 - B_1 > 1) = \left(\frac{1}{\sqrt{2\pi}} \int_2^\infty e^{-b_1^2/2} db_1\right) \left(\frac{1}{\sqrt{2\pi}} \int_1^\infty e^{-b_2^2/2} db_2\right)$$

Example:

1. Reflection at time s: For any $s \geq 0$ the process

$$\tilde{B}_t = \begin{cases} B_t & \text{if } t \leq s \\ B_s - (B_t - B_s) & \text{if } t > s \end{cases}$$

is a Brownian motion.

proof:

- $\tilde{B}_0 = 0$

- For $r \leq t$,

$$\begin{aligned} \text{Cov}(\tilde{B}_r, \tilde{B}_t) &= \begin{cases} \text{Cov}(B_r, B_t), & \text{if } r \leq t \leq s \\ \text{Cov}(B_r, B_s - (B_t - B_s)) & \text{if } r \leq s < t \\ \text{Cov}(B_s - (B_r - B_s), B_t - (B_t - B_s)) & \text{if } s < r \leq t \end{cases} \\ &= \begin{cases} r, & r \leq t \leq s \\ \text{Cov}(B_r, B_s) - \text{Cov}(B_r, B_t - B_s), & r \leq s < t \\ \text{Cov}(B_s, B_t) - \text{Cov}(B_s, B_t - B_s) + \text{Cov}(B_r - B_s, B_t - B_s), & s < r \leq t \end{cases} \\ &= \begin{cases} r, & r \leq t \leq s \\ r, & r \leq s < t \\ s + \text{Cov}(B_r, B_t - B_s) - \text{Cov}(B_s, B_t - B_s), & s < r < t \end{cases} \\ &= \begin{cases} r, & r \leq t \leq s \\ r, & r \leq s < t \\ s + r - s, & s < r < t \end{cases} \\ &= r. \end{aligned}$$

- $E[\tilde{B}_t] = 0$

- $\text{Var}[\tilde{B}_t] = t$

2. Scaling: For any $a > 0$, the process $Y_t = \frac{1}{\sqrt{a}} B_{at}$ is a Brownian motion.

proof:

$$\bullet Y_0 = \frac{1}{\sqrt{a}} B_{a \cdot 0} = \frac{1}{\sqrt{a}} B_0 = 0$$

$$\bullet \text{Cov}(Y_s, Y_t) = \text{Cov}\left(\frac{1}{\sqrt{a}} B_{as}, \frac{1}{\sqrt{a}} B_{at}\right) = \frac{1}{a} \text{Cov}(B_{as}, B_{at}) = \min(s, t)$$

• Clearly, $\frac{1}{\sqrt{a}} B_{at}$ is Gaussian because it is a linear transformation of a Gaussian.

3. Time Reversal: The process $Z_t = B_1 - B_{1-t}$ is a Brownian motion on $[0, 1]$.

proof:

$$\bullet Z_0 = B_1 - B_1 = 0$$

$$\begin{aligned} \bullet \text{Cov}(Z_t, Z_s) &= \text{Cov}(B_1 - B_{1-t}, B_1 - B_{1-s}) \\ &= \text{Cov}(B_1, B_1) - \text{Cov}(B_1, B_{1-s}) - \text{Cov}(B_{1-t}, B_1) + \text{Cov}(B_{1-t}, B_{1-s}) \\ &= 1 - (1-s) - (1-t) + \min\{1-t, 1-s\} \\ &= \begin{cases} t, & \text{if } t \leq s \\ s, & \text{if } s \leq t \end{cases} \\ &= \min(t, s) \end{aligned}$$