

Lecture #9: Conditioning on Several Random Variables

Recall: A σ -algebra $\mathcal{F}$ of a sample space Ω is a collection of events satisfying

Abstract

definition

of information

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. If $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

Trivial Examples:

1. $\{\emptyset, \Omega\}$ is a σ -algebra often denoted $\mathcal{F}_0$
2. $\{\emptyset, A, A^c, \Omega\}$ σ -algebra generated by set A .

Example: (Random Walk):

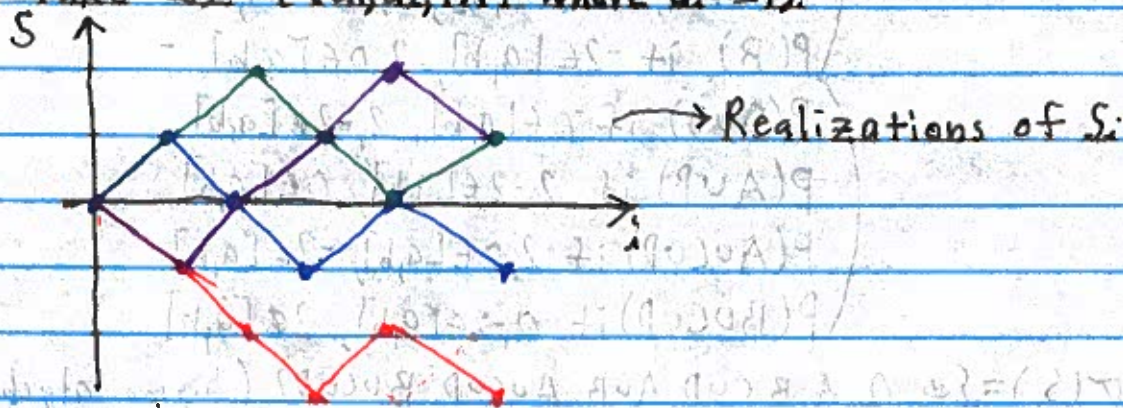
Let $X_i, i \in \mathbb{N}$ be a stochastic process satisfying

$$P(X_i = 1) = \frac{1}{2}, P(X_i = -1) = \frac{1}{2}$$

and define a stochastic process by

$$S_n = \sum_{i=1}^n X_i$$

We take $\Omega = \{\omega_1, \omega_2, \dots\}$ where $\omega_i = \pm 1$.



$$1. S_0 = 0$$

$$\Rightarrow P(S \in [a, b]) = P(\{\omega: 0 \in [a, b]\})$$

Asking Question

Information needed

$$\begin{cases} P(\emptyset) & \text{if } 0 \notin [a, b] \\ P(\Omega) & \text{if } 0 \in [a, b] \end{cases}$$

to ask question

The only events needed are $\{\emptyset, \Omega\} = \mathcal{F}_0$

2. $S_1 = 1$ or -1

$$P(S_1 \in [a, b]) = P(\{\omega : S_1(\omega) \in [a, b]\})$$

$$= \begin{cases} P(\emptyset) & \text{if } 1 \text{ and } -1 \notin [a, b] \\ P(\Omega) & \text{if } 1 \text{ and } -1 \in [a, b] \\ P(\{\omega : \omega = (1, \cdot, \cdot, \dots)\}) & \text{if } 1 \in [a, b] \text{ and } -1 \notin [a, b] \\ P(\{\omega : \omega = (-1, \cdot, \cdot, \dots)\}) & \text{if } -1 \in [a, b] \text{ and } 1 \notin [a, b] \end{cases}$$

The sigma algebra generated by S_1 is

$$\mathcal{F}_1 = \sigma(S_1) = \{\Omega, \emptyset, \{\omega : \omega = (1, \cdot, \cdot, \dots)\}, \{\omega : \omega = (-1, \cdot, \cdot, \dots)\}\}$$

Note,

$\mathcal{F}_0 \subset \mathcal{F}_1$ (New questions that can be asked include all of the old questions.)

3. $S_2 = 2, 0, -2$

$$\text{Let } A = \{\omega : \omega = (1, 1, \cdot, \cdot, \dots)\}, B = \{\omega : \omega = (-1, -1, \cdot, \cdot, \dots)\}$$

$$C = \{\omega : \omega = (-1, 1, \cdot, \cdot, \dots)\}, D = \{\omega : \omega = (1, -1, \cdot, \cdot, \dots)\}$$

$$P(S_2 \in [a, b]) = \begin{cases} P(\emptyset) & \text{if } 2, 0, -2 \notin [a, b] \\ P(\Omega) & \text{if } 2, 0, -2 \in [a, b] \\ P(A) & \text{if } 2 \in [a, b], 0, -2 \notin [a, b] \\ P(B) & \text{if } -2 \in [a, b], 2, 0 \notin [a, b] \\ P(C \cup D) & \text{if } 0 \in [a, b], 2, -2 \notin [a, b] \\ P(A \cup B) & \text{if } 2, -2 \in [a, b], 0 \notin [a, b] \\ P(A \cup C \cup D) & \text{if } 2, 0 \in [a, b], -2 \notin [a, b] \\ P(B \cup C \cup D) & \text{if } 0, -2 \in [a, b], 2 \notin [a, b] \end{cases}$$

$$\sigma(S_2) = \{\emptyset, \Omega, A, B, C \cup D, A \cup B, A \cup C \cup D, B \cup C \cup D\} \text{ (Sigma algebra generated by random variable)}$$

$$\mathcal{F}_2 = \sigma(S_i, i \in \{0, 1, 2\})$$

$$= \{\emptyset, \Omega, A, B, C, D, A \cup B, A \cup C, A \cup D, B \cup C, B \cup D, C \cup D, A \cup B \cup C, A \cup B \cup D, B \cup C \cup D, A \cup C \cup D\}$$

(Sigma algebra generated by stochastic process)

Definition - Let X be a random variable defined on $(\Omega, \mathcal{F}, P)$. Consider another σ -algebra $\mathcal{G} \subseteq \mathcal{F}$. Then X is said to be $\mathcal{G}$ measurable if and only if

$$\{\omega : X(\omega) \in [a, b]\} \in \mathcal{G} \text{ for all } b \in \mathbb{R}$$

Example:

In the previous example,

S_0 is $\mathcal{F}_0$, $\sigma(S_0) = \mathcal{F}_0$, $\sigma(S_1)$ and $\mathcal{F}_2$ measurable.

S_1 is $\sigma(S_0) = \mathcal{F}_1$, $\sigma(S_2)$ and $\mathcal{F}_2$ measurable.

S_2 is $\sigma(S_2)$ and $\mathcal{F}_2$ measurable.

Define $T_2 = S_0 + S_1 + S_2$. Then,

T_2 is $\mathcal{F}_2$ measurable.

Example:

Suppose $\mathcal{G} = \{\emptyset, \Omega\}$ and X is $\mathcal{G}$ measurable. Then, for all $[a, b]$,

$$\{\omega \in \Omega : X(\omega) \in [a, b]\} = \emptyset \text{ or } \Omega$$

$\Rightarrow X$ is a constant.

Example:

Suppose X is a random variable on $(\Omega, \mathcal{F}, P)$ and B a (Borel) set in $\mathbb{R}$. Define

$$Y = \mathbb{1}_{\{X \in B\}}$$

$$\Rightarrow \sigma(Y) = \{\emptyset, \{\omega : X(\omega) \in B\}, \{\omega : X(\omega) \in B^c\}, \Omega\} \subseteq \sigma(X)$$

$\Rightarrow Y$ is $\sigma(X)$ measurable.

Similarly, for a sequence of sets $B_1, B_2, \dots, B_n$, if we let

$$Z = \sum_{i=1}^n a_i \mathbb{1}_{\{X \in B_i\}}$$

then Z is $\sigma(X)$ measurable.

Consequence:

Y is $\sigma(X)$ measurable if and only if $Y = g(X)$ for some function g .

Definition: Let Y be an integrable random variable on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a σ -algebra of Ω . The conditional expectation of Y given $\mathcal{G}$, denote $E[Y|\mathcal{G}]$, is the $\mathcal{G}$ measurable random variable satisfying:

$$E[WY] = E[W E[Y|\mathcal{G}]] \quad (E[W(Y - E[Y|\mathcal{G}])] = 0).$$

for all W that are $\mathcal{G}$ measurable

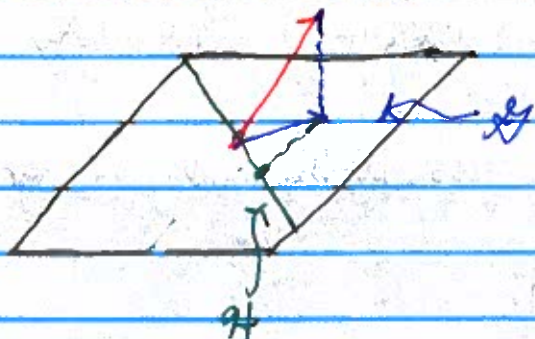
($E[Y|\mathcal{G}]$ is the best proxy for Y on the space of $\mathcal{G}$ measurable functions).

Notation:

$$E[Y|\sigma(X_1, \dots, X_n)] = E[Y|X_1, \dots, X_n]$$

Properties:

1. If Y is $\mathcal{G}$ measurable: $E[Y|\mathcal{G}] = Y$
2. If Y is $\mathcal{G}$ measurable: $E[YX|\mathcal{G}] = Y E[X|\mathcal{G}]$
3. If Y is independent of $\mathcal{G}$: $E[Y|\mathcal{G}] = E[Y]$
4. $E[aX + Y|\mathcal{G}] = a E[X|\mathcal{G}] + E[Y|\mathcal{G}]$
5. If $\mathcal{H} \subseteq \mathcal{G}$: $E[Y|\mathcal{H}] = E[E[Y|\mathcal{G}]|\mathcal{H}]$ (Tower Property)



Example:

$$\begin{aligned} 1. \mathbb{E}[e^{aB_1} | B_{1/2}] &= \mathbb{E}[e^{a(B_1 - B_{1/2})} e^{aB_{1/2}} | B_{1/2}] \\ &= e^{aB_{1/2}} \mathbb{E}[e^{a(B_1 - B_{1/2})} | B_{1/2}] \\ &= e^{aB_{1/2}} \mathbb{E}[e^{a(B_1 - B_{1/2})}] \\ &= e^{aB_{1/2}} e^{a^2/4} \\ &= e^{aB_{1/2} + a^2/4} \end{aligned}$$

The conditional density of $\mathbb{E}[B_1 | B_{1/2}] \sim N(B_{1/2}, 1/2)$

$$2. \mathbb{E}[e^{aB_{1/2}} | B_1]$$

$$\begin{aligned} \text{Note } \mathbb{E}[B_1(B_{1/2} - 1/2 B_1)] &= \mathbb{E}[B_1 B_{1/2}] - 1/2 \mathbb{E}[B_1^2] = 1/2 - 1/2 \cdot 1 = 0 \\ \Rightarrow B_{1/2} - 1/2 B_1 \text{ and } B_1 \text{ are independent.} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[e^{aB_{1/2}} | B_1] &= \mathbb{E}[e^{a(B_{1/2} - 1/2 B_1 + 1/2 B_1)} | B_1] = e^{1/2 a B_1} \mathbb{E}[e^{a(B_{1/2} - 1/2 B_1)} | B_1] \\ \mathbb{E}[e^{aB_{1/2}} | B_1] &= e^{1/2 a B_1} e^{a^2/8} \end{aligned}$$