

Problem 1. Let X_1 and X_2 be IID standard Gaussian random variables. Define

$$Z_1 = X_1 + 2X_2, \quad Z_2 = 2X_1 - X_2.$$

(a) Compute $\mathbb{E}[Z_1]$ and $\mathbb{E}[Z_2]$.

$$\mathbb{E}[Z_1] = \mathbb{E}[X_1] + 2\mathbb{E}[X_2] = 0$$

$$\mathbb{E}[Z_2] = 2\mathbb{E}[X_1] - \mathbb{E}[X_2] = 0$$

(b) Compute $\text{Var}(Z_1)$ and $\text{Var}(Z_2)$.

$$\begin{aligned} \text{Var}(Z_1) &= \mathbb{E}[(X_1 + 2X_2)^2] \\ &= \mathbb{E}[X_1^2] + 4\mathbb{E}[X_2^2] = 5 \end{aligned}$$

$$\begin{aligned} \text{Var}(Z_2) &= \mathbb{E}[(2X_1 - X_2)^2] \\ &= 4\mathbb{E}[X_1^2] + \mathbb{E}[X_2^2] = 5 \end{aligned}$$

(c) Compute $\text{Cov}(Z_1, Z_2)$.

$$\begin{aligned} \text{Cov}(Z_1, Z_2) &= \mathbb{E}[(X_1 + 2X_2)(2X_1 - X_2)] \\ &= 2\mathbb{E}[X_1^2] - 2\mathbb{E}[X_2^2] \\ &= 0 \end{aligned}$$

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$$\vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

(a) Compute the covariance matrix of $\vec{Z}$.

$$C = A \cdot A^T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

(b) Determine if Z_1 and Z_2 are independent.

$$\text{No, since } \text{Cov}(Z_1, Z_2) = C_{12} = 0.$$