

Problem 1. Let B_t be a standard Brownian motion and define

$$X_t = 2t + 3B_t, \quad t \geq 0.$$

(a) State the distribution of $B_t - B_s$ for $0 \leq s < t$.

$$B_t - B_s \sim \sqrt{t-s} N(0, 1)$$

(b) Compute $\mathbb{E}[X_2 - X_1]$ and $\text{Var}(X_2 - X_1)$.

$$\mathbb{E}[X_2 - X_1] = \mathbb{E}[4 + 3B_2 - 2 - 3B_1] = \mathbb{E}[2] + 3\mathbb{E}[B_2 - B_1] = 2$$

$$\mathbb{E}[(X_2 - X_1)^2] = \mathbb{E}[(2 + 3(B_2 - B_1))^2] = \mathbb{E}[4 + 6(B_2 - B_1) + 9(B_2 - B_1)^2] = 4 + 9 = 13$$

$$\Rightarrow \text{Var}(X_2 - X_1) = 13 - 4 = 9$$

Problem 1. Let B_t be a standard Brownian motion and define

$$X_t = 2t + 3B_t, \quad t \geq 0.$$

(a) State the distribution of $B_t - B_s$ for $0 \leq s < t$.

$$B_t - B_s \sim \sqrt{t-s} N(0, 1)$$

(b) Compute $\mathbb{E}[X_2 - X_1]$ and $\text{Var}(X_2 - X_1)$.

$$X_2 - X_1 = 2 + 3(B_2 - B_1)$$

$$\Rightarrow \mathbb{E}[X_2 - X_1] = 2$$

$$\mathbb{E}[(X_2 - X_1)^2] = \mathbb{E}[2 + 3(B_2 - B_1)]^2 = \mathbb{E}[4 + 6(B_2 - B_1) + 9(B_2 - B_1)^2] = 4 + 9 = 13$$

$$\text{Var}(X_2 - X_1) = 13 - 4 = 9.$$

(c) State the definition of $X_t \rightarrow X_1$ in L^2 as $t \rightarrow 1$. Then show that $X_t \rightarrow X_1$ in L^2 as $t \rightarrow 1$ (from either side).

$$X_t \rightarrow X_1 \text{ in } L^2 \text{ if and only if } \lim_{t \rightarrow 1} \mathbb{E}[(X_t - X_1)^2] = 0.$$

$$X_t - X_1 = 2(t-1) + 3(B_t - B_1)$$

$$\begin{aligned} \Rightarrow \mathbb{E}[(X_t - X_1)^2] &= \mathbb{E}[4(t-1)^2 + 12(t-1)(B_t - B_1) + 9(B_t - B_1)^2] \\ &= 4(t-1)^2 + 9(t-1) \end{aligned}$$

Consequently

$$\lim_{t \rightarrow 1} \mathbb{E}[(X_t - X_1)^2] = 0$$

Which implies $X_t \rightarrow X_1$ in L^2 as $t \rightarrow 1$.