Homework #1, Solutions

#1.

If $x\sin(\pi x) = \int_0^{x^2} f(t)\,dt$, where f is continuous, find $f(4)$.

Solution:

Differentiating it follows that

$$\begin{aligned}\frac{d}{dx}[x\sin(\pi x)] &= \sin(\pi x) + \pi x\cos(\pi x)\\ &= \frac{d}{dx}\int_0^{x^2} f(t)\,dt\\ &= \frac{d}{dx}[F(x^2) - F(0)]\\ &= 2x\,F'(x^2)\\ &= 2x\,f(x^2).\end{aligned}$$

Therefore,

$$2x\,f(x^2)\Big|_2 = \sin(\pi x) + \pi x\cos(\pi x)\Big|_2$$

$$\Rightarrow 4f(4) = \sin(2\pi) + 2\pi\cos(2\pi)$$

$$\Rightarrow f(4) = \pi/2.$$

#2.

Find the minimum area under the curve $y = 4x - x^3$ from $x = a$ to $x = a+1$.

Solution:

Let $g(a) = \int_a^{a+1}(4x - x^3)\,dx$. Therefore,

$$\begin{aligned}g'(a) &= 4(a+1) - (a+1)^3 - 4a + a^3\\ &= 4 - 3a^2 - 3a - 1\\ &= -3(a^2 + a - 1)\end{aligned}$$

Consequently,

$$g'(a) = 0$$

$$\Rightarrow a = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Furthermore,

$$g''(a) = -6a - 3$$

and therefore $a = -\frac{1+\sqrt{5}}{2}$ is where the minimum is obtained.

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#6.

If $f(x) = \int_0^x x^2 \sin(t^2)\,dt$, find $f'(x)$.

Solution:

$$f(x) = x^2 \int_0^x \sin(t^2)\,dt$$

$$\Rightarrow f'(x) = 2x \int_0^x \sin(t^2)\,dt + x^2 \sin(x^2).$$

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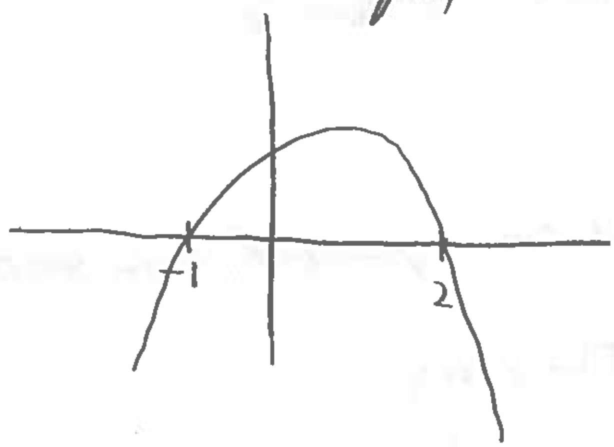
#7.

Find the interval $[a,b]$ for which $\int_a^b (2+x-x^2)\,dx$ is a maximum.

Solution:

Let $f(x) = 2+x-x^2$. The graph of $f(x)$ is given below:



So, the area is maximized if $a=-1$, $b=2$.

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#10

Find $\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin(t)}\sqrt{1+u^4}\,du\right)dt.$

Solution:

$$\frac{d^2}{dx^2}\int_0^x\left[\int_1^{\sin(t)}\sqrt{1+u^4}\,du\right]dt=\frac{d}{dx}\int_1^{\sin(x)}\sqrt{1+u^4}\,du$$

$$=\cos(x)\sqrt{1+\sin^4(x)}.$$