pg. 771, #2

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is known to be convergent.

a.) If $a_n > b_n$ what can you say about $\sum a_n$?

b.) If $a_n < b_n$ what can you say about $\sum a_n$?

Solution:

a.) This proves nothing.

b.) $\sum a_n$ converges.

pg. 771, #32

Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

Solution:

First,

$$\lim_{n\to\infty} \frac{1}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n n^{1/n}}.$$

Let $y = n^{1/n}$. Then,

$$\ln(y) = \frac{1}{n}\ln(n)$$

$$\Rightarrow \lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{1}{n}\ln(n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} y = e^0 = 1.$$

Therefore, for large n:

$$\frac{1}{n^{1+1/n}} \approx \frac{1}{n}.$$

This suggests comparing with $b_n = 1/n$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n \cdot n^{1/n}} \cdot n = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1.$$

Therefore, since $\sum b_n$ diverges it follows that $\sum a_n$ diverges as well.

pg. 772, #44

Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1+a_n)$ is convergent.

Solution:

First, let $f(x) = \ln(1+x)$ and $g(x) = x$. Then, if $x > 0$ it follows that

$$f'(x) = \frac{1}{1+x} < 1 = g'(x).$$

Moreover, $f(0) = g(0) = 0$. Therefore, since $a_n > 0$ it follows that

$$\sum_{n=1}^{\infty} \ln(1+a_n) < \sum_{n=1}^{\infty} a_n$$

Since $f(x) \leq g(x)$ if $x \geq 0$. Therefore, $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges. ∎

pg. 776, #4

Test the series for convergence.

$$\frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \cdots$$

Solution:

Consider the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+2)}$$

Let $a_n = \frac{1}{\ln(n+2)}$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+2)}{\ln(n+3)} < \frac{\ln(n+3)}{\ln(n+3)} = 1.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+2)}$$

converges. ∎

pg. 776, #16

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$$

Solution:

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{2^n}.$$

Let $a_n = \frac{n}{2^n}$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2 \cdot n} < \frac{n+n}{2n} = 1.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$$

converges.