pg. 783, #6

Determine whether the following series is absolutely convergent conditionally convergent: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$.

Solution:

Let $a_n = (-1)^{n-1} \frac{n}{n^2+4}$. Therefore, $|a_n| = \frac{n}{n^2+4}$ and

$$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{1}{n} \frac{n^2+4}{n} = 1.$$

Consequently, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges it follows from limit comparison test that $\sum |a_n|$ diveges. However, $\frac{n}{n^2+4}$ is decreasing with respect to n and thus by alternating series test $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ converges. Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$ converges conditionally. $\blacksquare$

pg. 783, #34

Determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{n\, 5^{2n}}{10^{n+1}}.$$

Solution:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)\, 5^{2(n+1)}}{10^{n+2}} \cdot \frac{10^{n+1}}{n\, 5^{2n}} = \lim_{n\to\infty} \frac{25}{10} \cdot \frac{n+1}{n} = \frac{5}{2}.$$

Therefore, by the ratio test this series diverges. $\blacksquare$

pg. 786, #24

Determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} n \sin(1/n).$$

Solution:

$$\lim_{n\to\infty} n \sin\left(\tfrac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Therefore, $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$ diverges. $\blacksquare$

pg. 786, #28

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}.$$

Solution:

$$\lim_{n\to\infty} \frac{1}{n^2}\cdot a_n = \lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{n^2}{e^{1/n}} = 1.$$

Therefore, by the limit comparison test since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges it follows that $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges.

pg. 786, 32

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

Solution!

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[(n+1)!]^{n+1}\, n^{4n}}{[n!]^n\, n^{4n+4}} = \lim_{n\to\infty} \frac{[(n+1)!]^n\cdot(n+1)!}{[n!]^n\, n^4}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^n\cdot(n+1)!}{n^4} = \infty.$$

Therefore, this series diverges.