pg. 791, #16

Find the radius of convergence and interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n = f(x).$$

Solution:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(2(n+1)-1)2^{n+1}}(x-1)^{n+1}\cdot\frac{(2n-1)\cdot 2^n}{(x-1)^n}\right|$$

$$\cong \lim_{n\to\infty}\left|\frac{x-1}{2}\right|.$$

Therefore, the series converges if

$$\left|\frac{x-1}{2}\right|<1$$

$$\Rightarrow |x-1|<2$$

$$\Rightarrow -2<x-1<2$$

$$\Rightarrow -1<x<3.$$

Therefore, the radius of convergence is 2. Now,

$$f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{(2n-1)2^n} = \sum_{n=1}^{\infty}\frac{1}{2n-1},$$

which diverges. Furthermore,

$$f(3) = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$$

which converges by alternating series test. Therefore, the interval of convergence is:

$$-1<x\leq 3.$$

pg. 791, #30

Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges when $x=-4$ and diverges when $x=6$, What can be said about the convergence or divergence of the following series?

Solution:

$\sum_{n=0}^{\infty} c_n$: converges

$\sum_{n=0}^{\infty} c_n 8^n$: diverges

$\sum_{n=0}^{\infty} c_n (-3)^n$: converges.

$\sum_{n=0}^{\infty} (-1)^n c_n 9^n$: diverges.

pg. 797, #6

Find a power series representation for the function and determine the interval of convergence.

Solution:

$$f(x) = \frac{4}{2x+3} = \frac{4}{3+2x} = \frac{4}{3\left(1+\frac{2x}{3}\right)} = \sum_{n=0}^{\infty} \frac{4}{3} (-1)^n \cdot \left(\frac{2x}{3}\right)^n$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{2^2 \cdot 2^n}{3 \cdot 3^n} \cdot (-1)^n \cdot x^n = \sum_{n=0}^{\infty} \frac{2^{n+2}(-1)^n}{3^{n+1}} x^n,$$

which converges for

$$\left|\frac{2}{3}x\right| < 1$$

$$\Rightarrow -\frac{3}{2} < x < \frac{3}{2}.$$

pg. 797, #14.

Find a power series representation for

a.) $\ln(1-x)$

b.) $x\ln(1-x)$

c.) $\ln(2)$.

Solution:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow -\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \ln(1-x) = \sum_{n=0}^{\infty} \frac{-x^{n+1}}{n+1}.$$

Therefore,

$$x\ln(1-x) = \sum_{n=0}^{\infty} \frac{-x^{n+2}}{n+1},$$

Furthermore,

$$\ln\left(1-\tfrac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{-1\left(\frac{1}{2}\right)^{n+1}}{n+1}$$

$$\Rightarrow \ln(1) - \ln(2) = \sum_{n=0}^{\infty} \frac{-1\cdot\left(\frac{1}{2}\right)^{n+1}}{n+1}$$

$$\Rightarrow \ln(2) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)}.$$

pg. 797, #16

Find a power series for:

$f(x) = x^2\tan^{-1}(x^3)$.

Solution:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

$$\Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

$$\Rightarrow f(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+5}}{2n+1}$$

pg. 798, #34

Show that

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1-\frac{x^2}{2}+\frac{x^4}{24}-\ldots$$

is a solution to the differential equation:

$$f''(x)+f(x)=0.$$

Solution:

$$f'(x)=\sum_{n=0}^{\infty}\frac{(-1)^n 2n x^{2n-1}}{(2n)!}=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n-1)!}$$

$$f''(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2n\cdot(2n-1)x^{2n-2}}{(2n)!}=\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n-2}}{(2n-2)!}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!}$$

Therefore,

$$f(x)+f''(x)=\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n}}{(2n)!}+\frac{(-1)^{n+1}x^{2n}}{(2n)!}\right]=0.$$