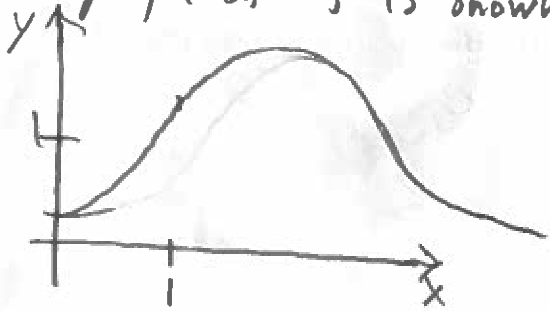
Homework #14

pg. 811, #2.

The graph of f is shown below:



a.) Explain why the series

$$1.6 - .8(x-1) + .4(x-1)^2 - .1(x-3)^3 + \cdots$$

is not the Taylor series of f about 1.

b.) Explain why the series

$$2.8 + .5(x-2) + 1.5(x-2)^2 - .1(x-2)^3 + \cdots$$

is not the Taylor series of f centered at 2.

Solution:

a.) $f'(1) > 0$, while the proposed Taylor series has a negative first derivative at 1.

b.) $f''(1) < 0$, while the Taylor series of f about 1 has a positive second derivative at 1.

■

pg. 811, #22

Find a Taylor series for $f(x) = 1/x$ about $a = -3$

Solution:

$$f(-3) = -\frac{1}{3} \Rightarrow c_0 = -\frac{1}{3}$$

$$f'(x) = -\frac{1}{x^2} \Rightarrow f'(-3) = \frac{-1}{3\cdot 3} \Rightarrow c_1 = \frac{-1}{3\cdot 3}$$

$$f''(x) = \frac{2}{x^3} \Rightarrow f''(-3) = \frac{-2}{3\cdot 3\cdot 3} \Rightarrow \frac{\cancel{-}2}{3\cdot 3\cdot 3\cdot 2!}$$

$\vdots$

$$f^{(n)}(-3) = \frac{(-1)^{n+1}}{3^n} \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{-\cancel{(-1)^{n+1}}}{3^{n+1}}(x+3)^n$$

■

pg. 811, #26

Find a Taylor series for $f(x)=\sqrt{x}$ about $a=16$.

Solution:

$$f(16)=\sqrt{16}=\sqrt{2^4}=2^2$$

$$f'(x)=\frac{1}{2}x^{-1/2} \Rightarrow f'(16)=\frac{1}{2}\cdot 2^{-2}=2^{-3}$$

$$f''(x)=\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow f'(16)=-\frac{1}{2^2}\cdot 2^{-6}=-2^{-8}$$

$$f'''(x)=\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \Rightarrow f'''(16)=3\cdot 2^{-13}$$

$$f^{iv}(x)=\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} \Rightarrow f^{iv}(16)=-5\cdot 3\cdot 2^{-18}$$

$$\Rightarrow f(x)=4+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{7-5n}\cdot(2n-1)(2n-3)\cdots 3\cdot 1}{n!}(x-16)^n$$

$$=4+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{7-5}(2n-1)!!}{n!}(x-16)^n$$

pg. 811, # 40

Find a Taylor series for the following function about $x=0$:

$$f(x) = x^2 \ln(1+x^3).$$

Solution:

$$x^2 \ln(1+x^3) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{3n+2}}{n}.$$

$\blacksquare$

pg. 812, # 54

Evaluate the indefinite integral as an infinite series:

$$\int x^2 \sin(x^2)\,dx.$$

Solution:

$$\int x^2 \sin(x^2)\,dx = \int x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}\,dx$$

$$\Rightarrow \int x^2 \sin(x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)!\,(2n+4)}.$$

$\blacksquare$

pg. 812, # 62

Use series to evaluate the following limit

$$\lim_{x\to 0} \frac{1-\cos(x)}{1+x-e^{-x}}$$

Solution:

$$\lim_{x\to 0} \frac{1-\cos(x)}{1+x-e^{x}} = \lim_{x\to 0} \frac{1-1+\frac{x^2}{2}-\frac{x^4}{24}+\ldots}{1+x-1-x-\frac{x^2}{2}-\frac{x^3}{6}+\ldots} = -1.$$

$\blacksquare$

pg. 812, # 812

If $f(x) = (1+x^3)^{30}$, what is $f^{(58)}(0)$.

Solution:

$$(1+x^3)^{30} = 1 + c_1 x^3 + c_2 x^6 + c_3 x^9 + \ldots + c_{30} x^{90}$$

Since 58 is not divisible by 3 it follows that $f^{(58)}(0) = 0$.

$\blacksquare$