Homework #6

pg. 481, #20

Prove that
$$\frac{d}{dx}[\sec^{-1}(x)] = \frac{1}{x\sqrt{x^2-1}}.$$

proof:

$$\sec(\sec^{-1}(x)) = x$$

$$\Rightarrow \sec(\sec^{-1}(x))\tan(\sec^{-1}(x))\cdot[\sec^{-1}(x)]' = 1$$

Let $\theta = \sec^{-1}(x)$. Then $\sec(\theta) = x$ and we can represent this angle using the following triangle:

Therefore, $\tan(\sec^{-1}(x)) = \tan(\theta) = \sqrt{1-x^2}$. Consequently,
$$[\sec^{-1}(x)]' = \frac{1}{\sec(\sec^{-1}(x))\tan(\sec^{-1}(x))} = \frac{1}{x\sqrt{1-x^2}}$$

$\blacksquare$

pg. 481, #34

Find the derivative of the function
$$y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\left(\sqrt{\frac{x-a}{x+a}}\right)$$

Solution:

$$y = \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\ln\left(\frac{x-a}{x+a}\right) = \tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\ln(x-a) - \frac{1}{2}\ln(x+a)$$

$$\Rightarrow y' = \frac{1}{a}\frac{1}{1+\frac{x^2}{a^2}} + \frac{1}{2(x-a)} - \frac{1}{2(x+a)}$$

$$= \frac{a}{a^2+x^2} + \frac{4a}{2(x^2-a^2)}$$

$$= \frac{a}{a^2+x^2} + \frac{2a}{(x^2-a^2)}.$$

$$\frac{a(x^2-a^2)+2a(a^2+x^2)}{x^4-a^4}$$

$$a^3+3ax^2$$

$\blacksquare$

pg. 482, #64

Calculate

$$\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx.$$

Solution:

$$\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\,dx = -\int_1^0 \frac{1}{1+u^2}\,du = \tan^{-1}(u)\Big|_0^1 = \pi/4.$$

$\blacksquare$

pg. 489, #18

Prove that

$$\frac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}.$$

Solution:

$$\frac{1+\tanh(x)}{1-\tanh(x)} = \frac{1+\frac{e^x-e^{-x}}{e^x+e^{-x}}}{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}} = \frac{e^x+e^{-x}+e^x-e^{-x}}{e^x+e^{-x}-e^x+e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x}.$$

$\blacksquare$

pg. 491, #66

Calculate

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt$$

Solution:

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt = \int_0^1 \frac{1}{\sqrt{(4t)^2+1}}\,dt$$

Let $u = 4t \Rightarrow du = 4dt$. Therefore,

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt = \frac{1}{4}\int_0^4 \frac{1}{\sqrt{1+u^2}}\,du = \frac{1}{4}\sinh^{-1}(4).$$

$\blacksquare$