Homework #9

pg. 744, 36

$$a_n = \frac{(-1)^{n+1} n}{n + n^{1/2}}.$$

Solution:

Let $b_1 = a_1$, $b_2 = a_3$, $b_3 = a_5$, $\vdots$, $b_n = a_{2n-1}$, $c_1 = a_2$, $c_2 = a_4$, $c_3 = a_6$, $\vdots$, $c_n = a_{2n}$

Therefore,

$$b_n = \frac{(-1)^{(2n-1)+1}(2n-1)}{(2n-1)+(2n-1)^{1/2}} = \frac{(-1)^{2n}(2n-1)}{(2n-1)+(2n-1)^{1/2}} = \frac{1}{1+(2n-1)^{-1/2}}$$

$$\Rightarrow \lim_{n\to\infty} b_n = 1.$$

$$c_n = \frac{(-1)^{2n+1}(2n)}{2n+(2n)^{1/2}} = \frac{(-1)^{2n+1}}{1+(2n)^{-1/2}} = \frac{-1}{1+(2n)^{-1/2}}$$

$$\Rightarrow \lim_{n\to\infty} c_n = -1.$$

Since,

$$\lim_{n\to\infty} b_n \neq \lim_{n\to\infty} c_n$$

it follows that a_n is divergent. ■

pg. 744, #48

$$a_n = \sqrt[n]{n}.$$

Solution:

Let $f(x) = x^{1/x}$. Then,

$$\lim_{x\to\infty} \ln(f(x)) = \lim_{x\to\infty} \frac{\ln(x)}{x}$$

$$= \lim_{x\to\infty} \frac{1}{x}$$

$$= 0$$

$$\Rightarrow \lim_{x\to\infty} f(x) = 1.$$

Therefore, since

$$a_n = f(n)$$

it follows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = 1.$$

pg. 745, #56

$$a_n = \frac{(-3)^n}{n!}$$

Solution:

Let $b_n = |a_n|$. Then,

$$b_n = \frac{|(-1)^n 3^n|}{|n!|} = \frac{3^n}{n!}$$

Now,

$$\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1}.$$

Consequently, b_n is decreasing for $n \geq 2$. Also, since $b_n \geq 0$ it follows that b_n is bounded. Moreover,

$$b_n = \frac{3^n}{n!} = \frac{3\cdot 3}{1\cdot 2} \cdot \frac{3\cdot 3\cdot 3 \cdots 3}{3\cdot 4\cdot 5\cdots n} \leq \frac{3\cdot 3}{1\cdot 2} \cdot \frac{\overbrace{3\cdot 3\cdots 3}^{n-1 \text{ times}}}{3\cdot 3\cdots 3} \cdot \frac{3}{n} = \frac{27}{2} \cdot \frac{1}{n},$$

Consequently,

$$\lim_{n\to\infty} b_n = 0$$

Therefore, since $|a_n| = b_n$ it follows that $a_n \to 0$.

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pg. 745, #74

Determine if the following sequence is monotonic:

$$a_n = \frac{1-n}{2+n}.$$

Solution:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1-(n+1)}{2+(n+1)} \cdot \frac{2+n}{1-n}\right| = \left|\frac{-n}{2+n+1} \cdot \frac{2+n}{1-n}\right| = \frac{2n+n^2}{(2+n+1)(n-1)}$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{2n+n^2}{(2+n+1)n} = \frac{2+n}{2+n+1} < 1.$$

Therefore, this sequence is montonic.

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pg. 745, #77

Determine if the following sequence is monotonic:

$$a_n = 3 - 2n e^{-n}.$$

Solution:

$$\frac{a_{n+1}}{a_n} = \frac{3-2(n+1)e^{-(n+1)}}{3-2ne^{-n}} > \frac{3-2ne^{-n}}{3-2ne^{-n}} = 1.$$

Therefore, a_n is monotonicly increasing.

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