Lecture: Rob

Series:

Recall that
$$\frac{1}{3} = .33\overline{3}$$

Alternatively, this means
$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$
$$= 3(.1) + 3(.1)^2 + 3(.1)^3 + \dots$$
$$= \sum_{k=1}^{\infty} 3(.1)^k.$$

What does this mean? It is hard enough to add two numbers together. Now we want to add infinite number of numbers together.

Define for any sequence $\{a_n\}$ a new sequence of partial sums by:

$$\boxed{S_1 = \sum_{k=1}^{1} a_k = a_1}$$
$$S_2 = \sum_{k=1}^{2} a_k = a_1 + a_2$$
$$S_3 = \sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = \sum_{k=1}^{n} a_k = \underbrace{a_1 + a_2 + \dots + a_{n-1}}_{S_{n-1}} + a_n = S_{n-1} + a_n$$

example:

$$a_n = \frac{1}{2^n}$$

$$S_1 = 1/2$$
$$S_2 = 1/2 + 1/4 = 3/4$$
$$S_3 = 3/4 + 1/8 = 7/8$$
$$S_4 = 7/8 + 1/16 = 15/16$$
$$\vdots$$
$$S_n = \frac{2^{n-1}-1}{2^{n-1}} + \frac{1}{2^n} = \frac{2\cdot(2^{n-1}-1)+1}{2^n} = \frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$$

$$\Rightarrow S_n = 1 - \frac{1}{2^n}$$

$$\Rightarrow \lim_{n\to\infty} S_n = 1.$$

Definition: If the sequence $S_n = \sum_{k=1}^{n} a_k$ converges to some number S, then we say the series defined by $\sum_{k=1} a_k$ converges to S. We write

$$\boxed{\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} S_n = S.}$$

example:

Does the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \ ?$$

partial fractions.

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\Rightarrow \lim_{n \to \infty} S_n = 1$$

Therefore,

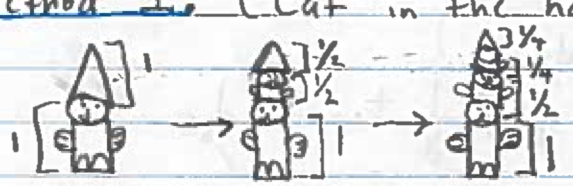
$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1.$$

example:

Does the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

a.) Method 1: (Cat in the hat)



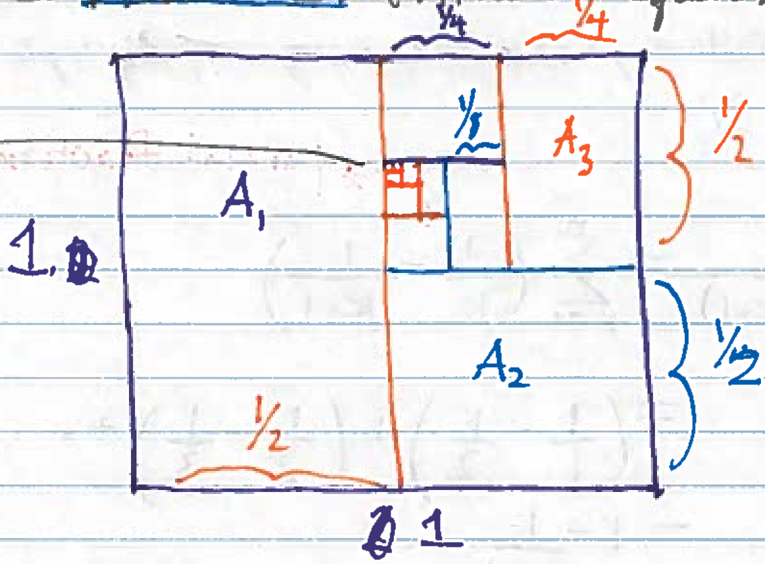
Total Length = 2

Sum of length of cats = $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$$\Rightarrow 2 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}$$

2. Method 2: (Area of Square)

Keep going forever!



$\cancel{A_1 = 1}$
$\cancel{A_2 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 = 1.}$
$\cancel{A_3 = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot \frac{1}{2} + \frac{1}{2}\cdot \frac{1}{2}}$

$A_1 = 1/2$
$A_2 = 1/4$
$A_3 = 1/8$
$A_4 = 1/16$
$A_5 = 1/32$

$$\Rightarrow A_1 + A_2 + A_3 + \ldots = 1$$
$$\Rightarrow 2A_1 + 2A_2 + 2A_3 + \ldots = 2$$
$$\Rightarrow 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$
$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2.$$

3. Method 3: (Analytic)

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}$$
$$\Rightarrow 2S_n = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{n-1}} + \frac{1}{2^n}$$
$$\Rightarrow 2S_n = 2 + S_n + \frac{1}{2^n}$$
$$\Rightarrow S_n = 2 + \frac{1}{2^n}$$
$$\Rightarrow \lim_{n\to\infty} S_n = 2.$$

Theorem - For $a \neq 0$, the geometric series $\sum_{k=1}^{\infty} ar^{k-1}$ converges to $\frac{a}{1-r}$ if $|r|<1$ and diverges if $|r| \geq 1$.

proof:

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$\Rightarrow rS_n = ar + ar^2 + ar^3 + \dots + ar^n$$

$$= -a + a + ar + ar^2 + ar^3 + \dots + ar^n$$

$$= -a + S_n + ar^n$$

$$\Rightarrow (r-1)S_n = -a + ar^n$$

$$\Rightarrow S_n = \frac{a}{1-r} + \frac{a}{r-1} \cdot r^n$$

Therefore,

1. If $|r|<1$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

2. If $|r|>1$

$\lim_{n\to\infty} S_n$ does not exist.

3. If $r=1$, then

$$S_n = a + a + a + \dots + a$$
$$= na$$

Which diverges.

example:

Is the series $\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n}$ divergent or convergent?

$$\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^{-(n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}}$$

$$= \sum_{n=1}^{\infty} 4 \cdot \frac{4^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{3}\right)^{n-1} = 4 \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^{n-1}$$

$\Rightarrow$ This summation diverges.

example:

If $|x|<1$ what is

$$\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$$

change summation.

This is amazing!! A function can be represented by a series!!! Gives a systematic way to approximate a function!!

Test for divergence - If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

example:

1. $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$. $\lim_{n\to\infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \Rightarrow$ This series diverges.

2. $$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$= 3 \cdot 1 + 1 = 4.$$

If $\lim_{n\to\infty} a_n = 0$ does $\sum_{n=1}^{\infty} a_n$ converge?

example:

$$\sum_{i=1}^{n} \frac{1}{n}$$

Important Fact!!!

$$\sum_{i=1}^{n} \frac{1}{n} = \text{Area of rectangles} > \int_1^n \frac{1}{x}\,dx = \ln(n) \to \infty$$

$$\Rightarrow \sum_{i=1}^{n} \frac{1}{n} = \infty.$$