Lecture: Staci

Sequences

Definition: A sequence is

1. (Version 1) – A list of numbers with a definite order:

$$\underset{\substack{\uparrow\\ \text{first}\\ \text{term}}}{a_1},\ \underset{\substack{\uparrow\\ \text{second}\\ \text{term}}}{a_2},\ \dots,\ \underset{\substack{\uparrow\\ \text{n-th}\\ \text{term}}}{a_n},\ \dots$$

Notation:
$\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

2. (Version 2) – Any function whose domain is the set of positive integers:

$$f(n) = a_n.$$

stick in number which is a positive integer

out comes any real number.

Notation example:

1) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$, $a_n = \dfrac{n}{n+1}$, $\left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dots\right\}$

example:

Let $a_n = \dfrac{n+1}{n}$. Then,

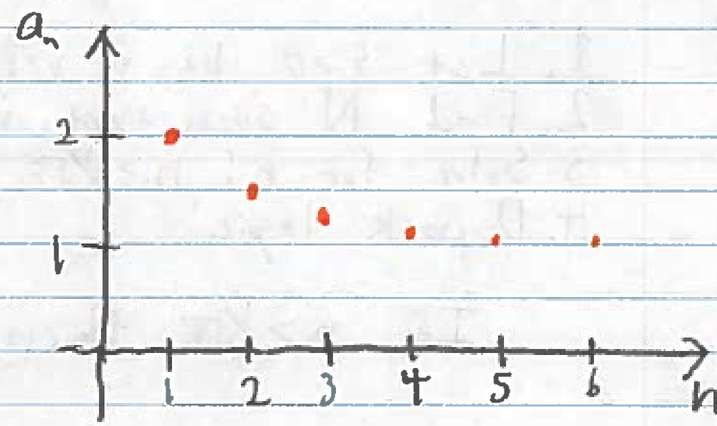
$$a_1 = \frac{2}{1} = 2$$

$$a_2 = \frac{2+1}{2} = \frac{3}{2}$$

$$a_3 = \frac{3+1}{3} = \frac{4}{3}$$

$$a_4 = \frac{4+1}{4} = \frac{5}{4}$$

$$\vdots$$



As n gets large a_n gets arbitrarily close to 1.

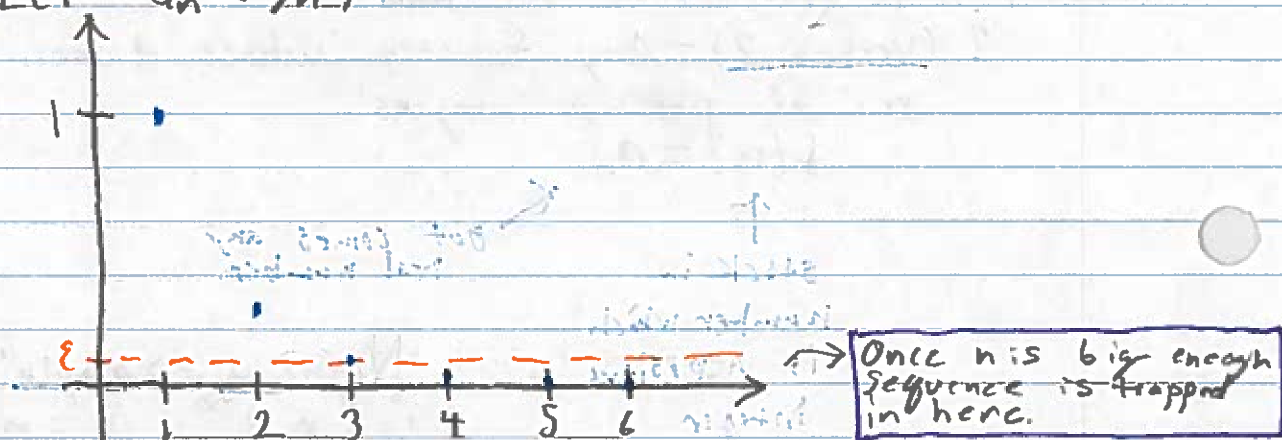
Definition - A sequence $\{a_n\}$ has limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if the terms a_n get arbitrarily close to L as n gets sufficiently large. If $\lim_{n\to\infty} a_n = L$ we say $\{a_n\}$ converges (is convergent). Otherwise, the sequence is divergent.

example (Proving a sequence converges):

Let $a_n = 1/n^2$.



It looks like $a_n \to 0$ as $n\to\infty$. How do we prove this?? We need to show a_n gets arbitrarily close to 0 as n gets large.

1. Let $\varepsilon \geq 0$ be a very small number.
2. Find N such that $n \geq N$ implies $1/n^2 < \varepsilon$.
3. Solve for n: $n > 1/\sqrt{\varepsilon}$.
4. Unpack 'logic'

$$\text{If } n > 1/\sqrt{\varepsilon} \text{ then } a_n = \frac{1}{n^2} < \varepsilon$$

Theorem:

Suppose that $\lim_{x\to\infty} f(x) = L$. Then, $\lim_{n\to\infty} f(n) = L$ also.

example:

1. $a_n = \frac{5n+2}{3n-5}$,

Let $f(x) = \frac{5x+2}{3x-5}$. Then, $\lim_{x\to\infty} f(x) = \frac{5}{3}$.

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \frac{5}{3}.$$

2. $a_n = \frac{n+1}{e^n}$.

Let $f(x) = \frac{x+1}{e^x}$. Then, $\lim_{x\to\infty} f(x) = 0$.

Therefore,

$$\lim_{n\to\infty} \frac{n+1}{e^n} = 0.$$

The converse of this theorem is not true!! That is $\lim_{n\to\infty} f(n) = L$ does not imply $\lim_{x\to\infty} f(x) = L$.

example:

Let $a_n = \cos(2\pi n)$. Then,

$a_1 = 1$, $a_2 = 1$, $a_3 = 1$, ...

$\Rightarrow \lim_{n\to\infty} a_n = 1$.

However,

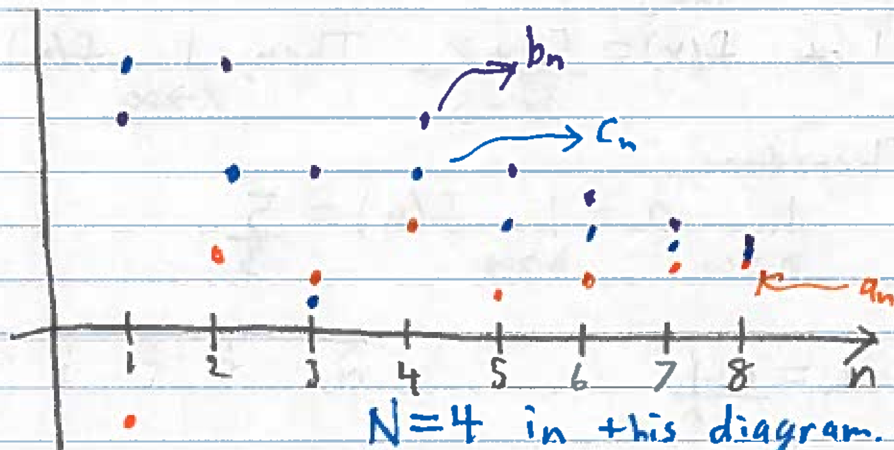
$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \cos(2\pi x)$ does not exist!

Squeeze Theorem - Suppose $\{a_n\}$, $\{b_n\}$ are convergent sequences with limit L. If there exists $N>0$ such that $n>N$ implies

$$a_n \le c_n \le b_n$$

then

$$\lim_{n\to\infty} c_n = L$$



$N=4$ in this diagram.

example:

Let $a_n = \frac{\sin(n)}{n^2}$.

$$-\frac{1}{n^2} < \frac{\sin(n)}{n^2} < \frac{1}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0.$$

Corollary: If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

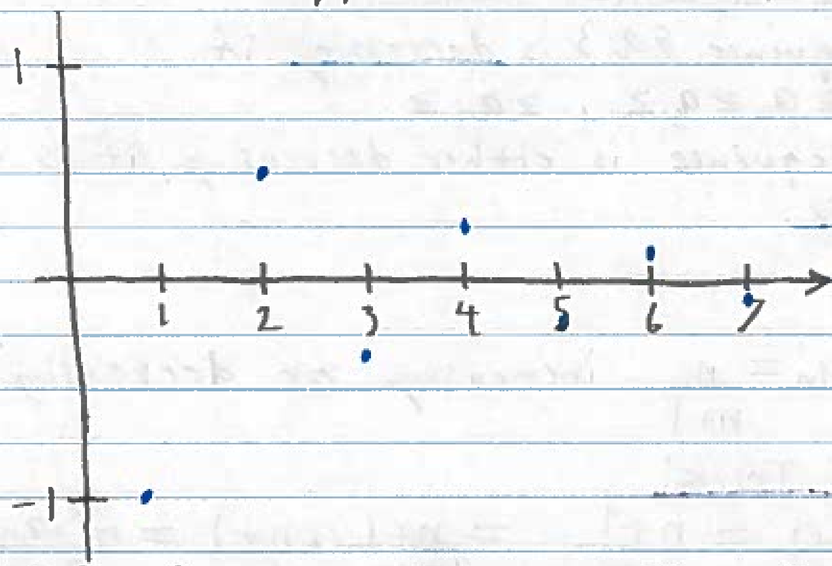
proof:

$$-|a_n| \le a_n \le |a_n|$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} |a_n| = 0 \quad \text{(Squeeze Theorem).}$$

example:

1. Let $a_n = \frac{(-1)^n}{n}$.



$|a_n| = \frac{1}{n} \to 0$ as $n \to \infty$. Therefore, $a_n \to 0$ as $n \to \infty$.

2. Let

$$a_n = \frac{n!}{n^n}$$

$$\boxed{\begin{array}{l}(\text{Recall: } n! = n(n-1)(n-2)\cdots 1.) \\ \quad 4! = 4\cdot 3\cdot 2\cdot 1 = 24 \\ \quad 0! = 1 \quad (\text{Definition}).\end{array}}$$

$$\begin{aligned} 0 < a_n &= \frac{n\cdot(n-1)(n-2)\cdots 2\cdot 1}{n\cdot n\cdot n\cdots n\cdot n} \\ &= \frac{1}{n}\cdot\frac{2\cdot 3\cdot 4\cdots n}{n\cdot n\cdot n\cdots n} \\ &\le \frac{1}{n}\cdot\frac{n\cdot n\cdot n\cdots n}{n\cdot n\cdot n\cdots n} \\ &= \frac{1}{n}. \end{aligned}$$

By Squeeze Theorem,

$$\lim_{n\to\infty} a_n = 0.$$

Definition–

1. The sequence $\{a_n\}$ is increasing if
$$a_1 \le a_2 \le a_3 \le \dots \le a_n \le \dots$$
2. The sequence $\{a_n\}$ is decreasing if
$$a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n \ge \dots$$

If a sequence is either decreasing, it is called monotonic.

example:

1. Is $a_n = \dfrac{n}{n+1}$ increasing or decreasing?

Useful Trick:
$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$$
$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n^2+2n}{n^2+2n} + \frac{1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1.$$
$$\Rightarrow \frac{a_{n+1}}{a_n} > 1 \Rightarrow a_{n+1} > a_n.$$

This sequence is increasing.

2. Is $\left\{\dfrac{n!}{e^n}\right\}$ decreasing or increasing?

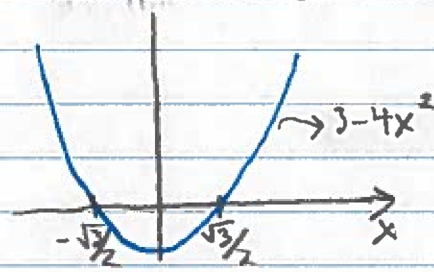
$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{e^{n+1}}}{\frac{n!}{e^n}} = \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} = \frac{(n+1)\cancel{n}\cdot\cancel{(n-1)}\cdots\cancel{2}\cdot\cancel{1}\cdot e^n}{e^{n+1}\cdot\cancel{n}\cdot\cancel{(n-1)}\cdot(n-2)\cdots\cancel{2}\cdot\cancel{1}}$$
$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n+1}{e}$$

If $n \ge 1$ we have
$$\frac{a_{n+1}}{a_n} > 1 \Rightarrow a_{n+1} \ge a_n. \text{ Increasing if } n > 2.$$

Definition - A sequence $\{a_n\}$ is bounded if there exists $M > 0$ for which $|a_n| \leq M$.

example:
Show that $a_n = \dfrac{3-4n^2}{n^2+1}$ is bounded.

Since $4n^2 - 3 > 0$ for $n \geq 1$ it follows that:



$$|a_n| = \left|\frac{3-4n^2}{n^2+1}\right| = \frac{4n^2-3}{n^2+1} < \frac{4n^2}{n^2+1} < \frac{4n^2}{n^2} = 4.$$

Therefore, a_n is bounded.

Theorem - Every bounded monotonic sequence has a limit.

Big result very useful for proving a sequence converges.

example:
Does $\left\{\dfrac{2^n}{n!}\right\}$ converge?

1. $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{n!}{2^n} = \dfrac{2}{n+1} \leq 1$ for $n \geq 1$.

$\Rightarrow a_n$ is decreasing (monotonic).

2. $|a_n| = \dfrac{2^n}{n!} = \dfrac{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}{n \cdot (n-1) \cdots 2 \cdot 1}$

$$\leq \frac{n}{n} \cdot \frac{n-1}{n-1} \cdot \ldots \cdot \frac{2}{2} \cdot \frac{2}{1}$$
$$= 2.$$

Since a_n is bounded and monotonic $\Rightarrow a_n$ converges.