Chapter 10: Iterated Maps

$f: \mathbb{R} \to \mathbb{R}$, $f: [0,1] \to [0,1]$, $f: S^1 \to S^1$

Discrete dynamical system:

$$x_{n+1} = f(x_n)$$

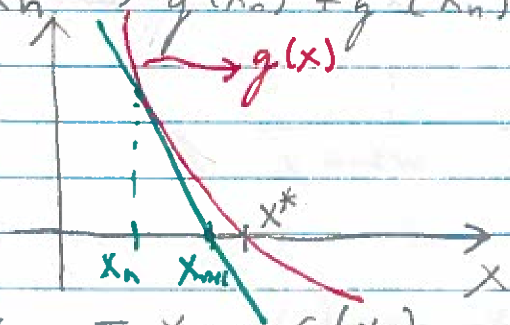
Newton's Method

Find root x^* of $g(x)$:

Given x_n find 'a' better approximation of x^*

Taylor expand:

$$x_n \Rightarrow g(x_n) + g'(x_n)(x - x_n)$$



$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

Definitions:

1. Orbit of x_0: $\gamma(x_0) = \{x_n : x_n = f(x_{n-1})\}$.
2. Fixed points: x is a fixed point if $x = f(x) \Rightarrow \gamma(x) = \{$
3. Period k orbits: $\gamma(x_0) = \{x_0, x_1, \ldots, x_k\}$.

Stability:

Let x^* be a fixed point of f and assume that f is differentiable near x^*. Then,

$$x^* \text{ is } \begin{cases} \text{stable} & \text{if } |f'(x^*)| < 1 \\ \text{unstable} & \text{if } |f'(x^*)| > 1. \end{cases}$$

proof:

Let δ_n be the sequence satisfying $\delta_n = x_n - x_{n-1}$, for $x_0 = x^* + \delta_0$. Then,

$$x_1 = f(x^* + \delta_0) = f(x^*) + f'(x^*)\delta_0 + O(\delta_0^2)$$

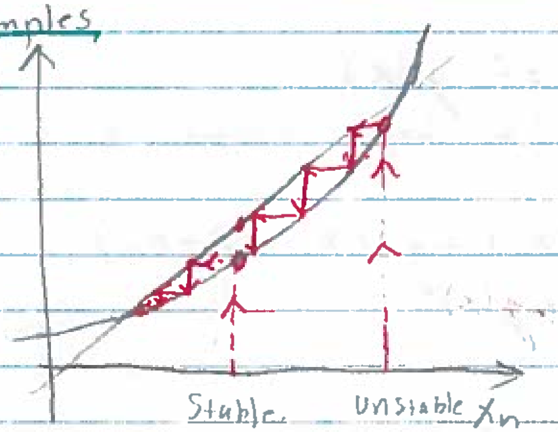
$$\Rightarrow \delta_1 = f'(x^*)\delta_0 + O(\delta_0^2)$$

$$\Rightarrow \delta_2 \approx f'(x^*)\delta_1 \approx f'(x)^2 \delta_0$$

$$\Rightarrow \delta_n = f'(x^*)^n \delta_0$$

$$\Rightarrow \delta_n \to \begin{cases} 0 & \text{if } |f'(x^*)| < 1 \\ \infty & \text{if } |f'(x^*)| > 1 \end{cases}$$

Examples



Example:

$$X_{n+1} = f(X_n) = \tfrac{3}{2} X_n (1 - X_n)$$

Fixed points:

$$x^* = \tfrac{3}{2} x^* (1 - x^*)$$

$$2x^* = 3x^* - 3x^{*2}$$

$$\Rightarrow \quad x^*(3x^* - 1)$$

$$\Rightarrow \quad x^* = 0, \; x^* = \tfrac{1}{3}$$

$$f'(x) = \tfrac{3}{2} - 3x$$

$f'(0) = \tfrac{3}{2} \Rightarrow 0$ is unstable

$f'(\tfrac{1}{3}) = \tfrac{3}{2} - 1 = \tfrac{1}{2} \Rightarrow \tfrac{1}{3}$ is stable

Period 2-orbits

$f(x) = -x^3, \quad x_{n+1} = f(x_n)$

Fixed points:

$x^* + x^{*3} = 0$

$\Rightarrow x^* = 0$

Stability: $f'(0) = 0 \Rightarrow$ stable

Period 2-orbits:

$f^2(x^*) = (f \circ f)(x^*) = x^*$ (fixed point for two iterations)

$\Rightarrow x^{*9} = x^*$

$\Rightarrow x^* = 0, \quad x^{*8} - 1 = 0$

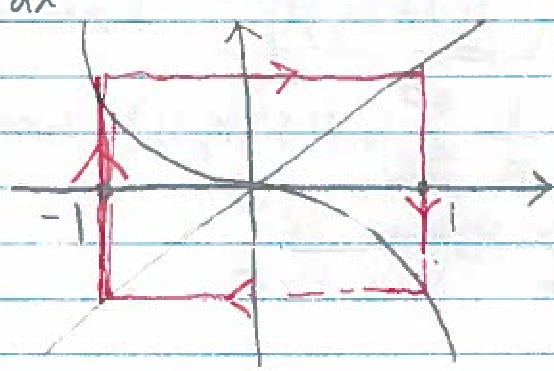
$\Rightarrow x^* = 0, \quad x^*_{1,2} = \pm 1$

Not really 2-orbit   This is the real 2-orbit

Stability of 2-orbit:

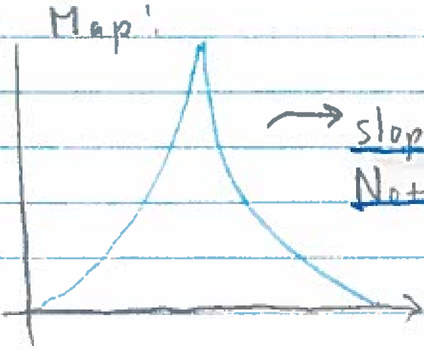
$\frac{d}{dx} f^2(x_1^*) = f'(f(x_1^*)) f'(x_1^*) = \underbrace{f'(x_2^*) f'(x_1^*)}_{\text{nice trick}}$

$\Rightarrow \frac{d}{dx} f^2(x_1^*) = (-3)(-3) = 9 \Rightarrow$ unstable $\Rightarrow$ product of slopes along periodic orbit



Example

Lorenz Map:



$\rightarrow$ slope > 1 everywhere

Not possible to have stable periodic orbits

Lyapunov Exponents

How to measure sensitivity to initial data?

1. Pick x_0 and compute orbit $\gamma(x_0) = \{x_0, x_1, \dots\}$
2. Pick y_0 close to x_0 and compute orbit $\gamma(y_0) = \{y_0, y_1, \dots\}$
3. Let $\delta_n = y_n - x_n$

$$\Rightarrow \delta_n = f(y_{n-1}) - f(x_{n-1})$$
$$= f(x_{n-1} + \delta_{n-1}) - f(x_{n-1})$$
$$\approx f'(x_{n-1})\,\delta_{n-1}$$
$$\Rightarrow \delta_n = f'(x_{n-1}) \cdots f'(x_0)\,\delta_0$$
$$= \underbrace{\left(\prod_{j=0}^{n-1} f'(x_j)\right)}_{\text{measure of seperation of nearby trajectories}} \delta_0$$

We expect

$$|\delta_n| = |\delta_0| e^{n\lambda(x_0)}$$
$$\Rightarrow e^{\lambda(x_0)} = \left(\prod_{j=0}^{n-1} |f'(x_j)|\right)^{1/n}$$

We define,

$$L(x_0) = \lim_{n\to\infty} \left(\prod_{j=0}^{n-1} |f'(x_j)|\right)^{1/n}, \quad \text{Lyapunov multiplier}$$

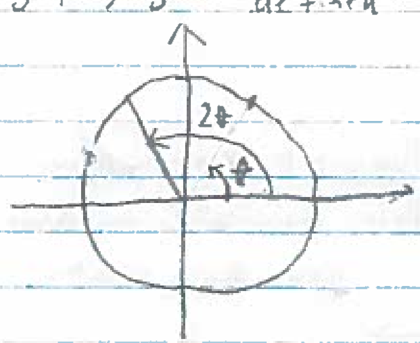
$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln(|f'(x_j)|), \quad \text{Lyapunov exponent}$$

Solutions exponentially seperate if

$$L(x_0) > 1 \quad \text{or} \quad \lambda(x_0) > 0$$

Example:

$f: S^1 \mapsto S^1$ defined by $\theta \mapsto 2\theta$



$$\delta_0 e^{\lambda n} = \delta_0 |f^{n\prime}(x^*)|$$
$$\lambda = \frac{1}{n} \ln(|f^{n\prime}(x^*)|)$$

1. Since $f'(\theta) = 2$ for all θ we have that
$$\lambda(\theta_0) = \ln(2) > 0$$
i.e. we have sensitive dependence on intial data

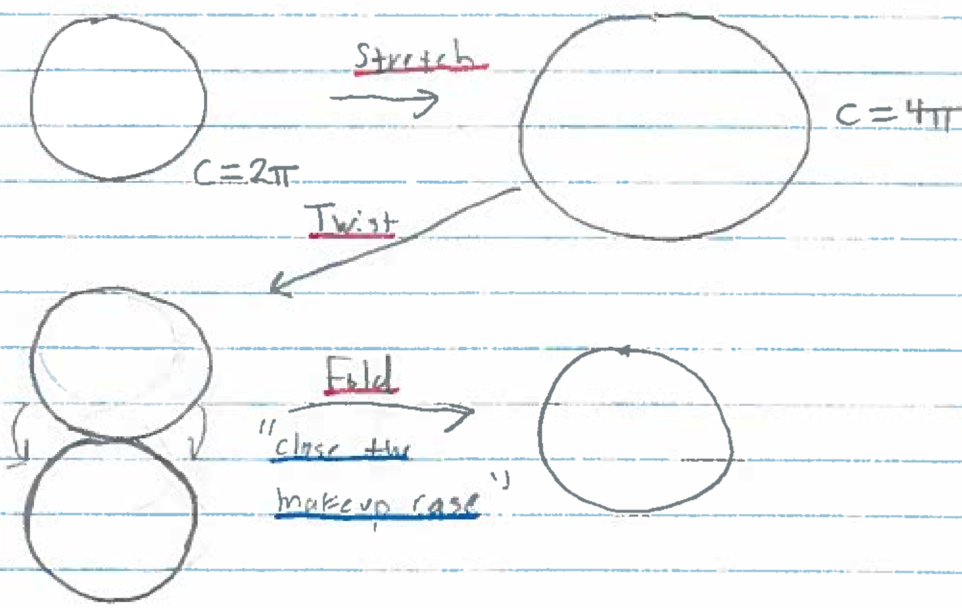
2. Let $P_k = \{\theta \in S^1 : f^k(\theta) = \theta\}$, i.e. set of not necessar smallest periodic orbits.

Let $P = \bigcup_{k=1}^{\infty} P_k$ → all periodic orbits.

Now, $\theta \in P$ if $2^k \theta = \theta + 2\pi n$ for some $n \in \mathbb{N}$.
$$\Rightarrow \theta = \frac{2\pi n}{2^k - 1} \quad \text{with} \quad 0 \leq n \leq 2^{k-1}$$
$$\Downarrow$$
* P is dense in S, meaning $\forall \varphi \in S$ and $\forall \varepsilon > 0$ $\exists \theta \in P$ such that $|\theta - \varphi| < \varepsilon$.

3. Topologically transitive. – The attractor is "indecomposable" [For any two open arcs $O_1, O_2 \subset S^1$, $\exists n > 0$ with $f^n(O_1) \cap O_2 \neq \emptyset$.]

Stretching, twisting and folding:



We say that f is chaotic if

- it has sensitive dependence on initial conditions
- the set of periodic orbits is dense.
- it is topologically transitive.