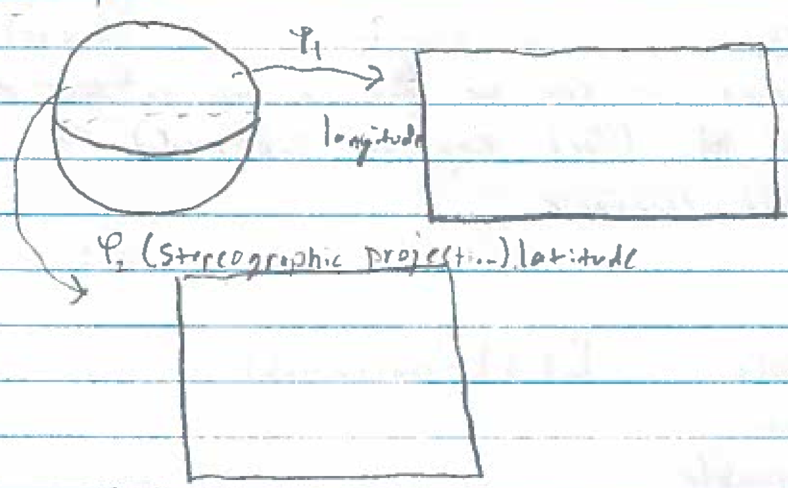
Chapter 11: Fractals

Dimension - How do we measure the dimension of a set?
One idea is to count the number of coordinates needed to describe set.

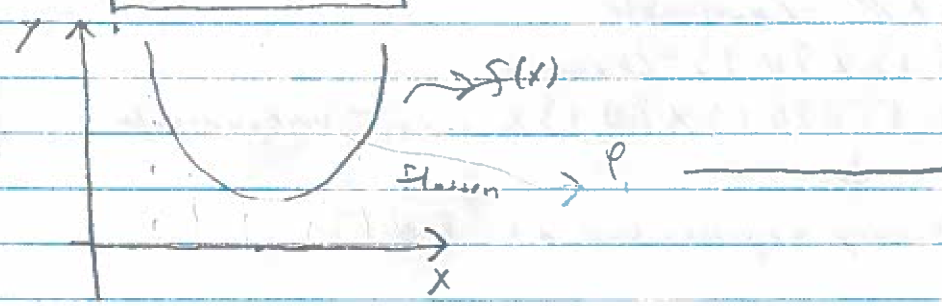
* A smooth manifold of dimension n is a set M^n that locally looks like $\mathbb{R}^n$. I.e, for each $p \in M^n$ there exists an open set O_p containing p and a smooth map $f: O_p \to \mathbb{R}^n$ with smooth inverse f^{-1}.
* $(f_p, O_p) \to$ Coordinate chart (This is like a map of the set)
* The collection of all coordinate charts is called an atlas.

What about non-smooth sets??

Example:



$\Rightarrow$ The sphere is two dimensional



Example:

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}, \quad \text{on interval } [0,1].$$

$$|f(x)| \leq \sum_{k=1}^{\infty} \frac{1}{\pi k^2} = \frac{\pi}{6} \Rightarrow f \text{ is continuous}$$

However:

$$f'(x) \overset{"="}{=} \sum_{k=1}^{\infty} \cos(\pi k^2 x)$$

For large k $\cos(\pi k^2 x) \approx 1$ i.o.

$$\Rightarrow |f'(x)| = \infty.$$

f is not differentiable almost everywhere.

$\Rightarrow$ Consequence: f has infinite arclength:

$$L = \int_0^1 \sqrt{1+f'(x)^2}\, dx = \infty$$

$\Rightarrow$ Consequence: We cannot define dimension in the classical sense

Size of Sets

Countable — finite or can be put in one to one correspondence with $\mathbb{N}$ (Set can be indexed).

uncountable — not countable.

Examples:

$\mathbb{Z}$ — countable $\qquad [0,1]$ — uncountable

$\mathbb{Q}$ — countable

$\mathbb{Z} \times \mathbb{Z}$ — countable.

$\{0,1\} \times \{0,1\}$ — countable

$\{0,1\} \times \{0,1\} \times \{0,1\} \times \ldots$ — uncountable

$\downarrow$

Binary representation of $[0,1]$

Sets of measure 0 - A set S has measure 0 if $\forall \varepsilon > 0$, S is a subset of a union of open cubes the sum of whose volume is less than ε.

Example:

$\mathbb{Q}$ is a set of measure 0.

proof:

We can index $\mathbb{Q}$ by points $\{r_1, r_2, \dots\}$. Let $b_i = \{r_i - \frac{\varepsilon}{2 \cdot 2^i}, r_i + \frac{\varepsilon}{2 \cdot 2^i}\}$

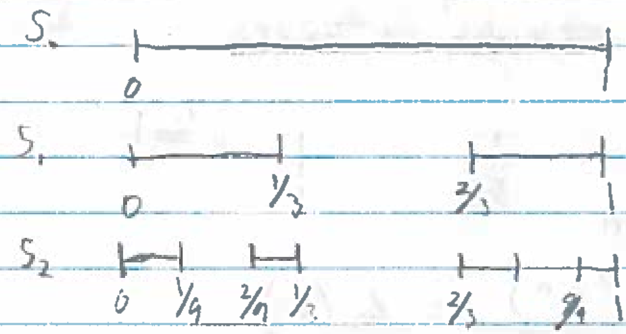
Then,

$$V\left(\bigcup_{i=1}^{\infty} b_i\right) \leq \sum_{i=1}^{\infty} V(b_i) = \sum_{i=1}^{\infty} \varepsilon/2^i \leq \pi^2/6 \cdot \varepsilon$$

Consequence: There are two ways to measure the size of a set.

Example: Cantor Set

The Cantor set is formed by removing middle third of sets:



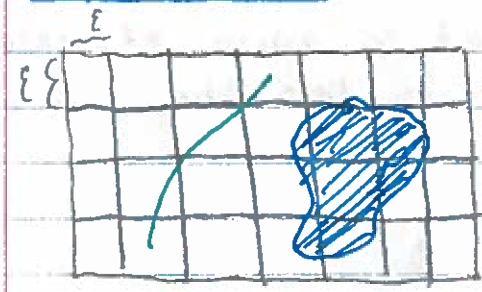
$\rightarrow$ 2^n intervals at each stage

Do this to ∞. $\longrightarrow S_\infty = $ Cantor Set.

$x \in S_\infty \Leftrightarrow x \in \bigcap_{n=1}^{\infty} S_n$

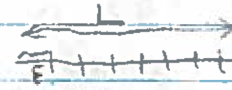
1. S_∞ is uncountable $\rightarrow$ can be put into correspondence with $[0,1]$ by binary representation
2. S_∞ has measure 0 $\rightarrow$ take balls of volume $\left(\frac{1}{3}\right)^n \cdot 2^n = \left(\frac{2}{3}\right)^n$ take limit $n \to \infty$

Box Dimension



Let $A \subset \mathbb{R}^n$. Take a mesh of boxes of length ε. Let $N(\varepsilon)$ be the number of boxes that intersect with A.

1-dim: $N(\varepsilon) \sim 1/\varepsilon$



$N \sim L/\varepsilon$

2-dim: $N(\varepsilon) \sim 1/\varepsilon^2$

$N \sim A/\varepsilon^2$

Assuming there is some scaling law
$N(\varepsilon) \sim 1/\varepsilon^d$

then

$$\boxed{d = \lim_{\varepsilon \to 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}, \quad d - \text{box dimension}}$$

Example:

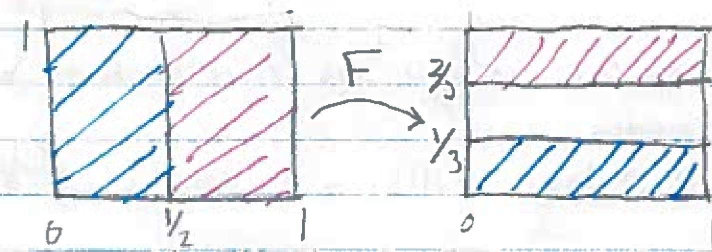
What is the box dimension of the Cantor set?
We can construct a sequence of coverings. Let
$\varepsilon_n = (1/3)^n \rightarrow$ width of boxes.

$n = 1$

then $N(\varepsilon) = 2^n$

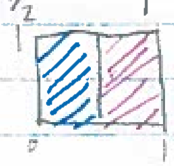
$$d = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)}$$

Example:



$$F(x,y) = \begin{cases} (2x, \, y/3) & 0 \le x \le 1/2 \\ (2x-1, \, y/3 + 2/3), & 1/2 \le x \le 1 \end{cases}$$

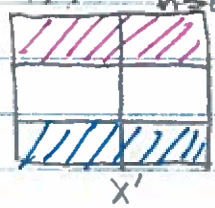
1. Squish:



2. Stretch:

3. Stack:

The planar baker's map is chaotic. What is its attracting set? $A = \bigcap_{n\geq 0} F^n([0,1] \times [0,1])$



Cross sections:

The attracting set

$A = [0,1] \times C \to C$ is the Cantor set

Box dimension is $1 + \frac{\ln(2)}{\ln(3)}$

Example:

What is the box dimension of $\mathbb{Q} \cap [0,1]$?

No matter how you cover this set $N(\varepsilon) = 1/\varepsilon \Rightarrow d = 1$.

Hausdorff Dimension

Intuition:

Cover a set with disks

$A \sim \pi r^{2} N$ → 2-dimensional measure.

$A \sim \frac{\pi}{N} \to 0$, However what if we changed the power?

$H \sim \pi r^1 N \sim \pi \to$ 1 dimensional object

Let $\Gamma(\varepsilon)$ be covering (set of coverings) of $A \subset \mathbb{R}^n$ of closed balls B_i of radii $r_i \leq \varepsilon$. For $\gamma \in \Gamma(\varepsilon)$ set

$$H_\alpha(A, \varepsilon) = \inf_{\gamma \in \Gamma(\varepsilon)} \sum r_i^\alpha$$

Hausdorff measure

$$H_\alpha(A) = \lim_{\varepsilon \to 0} H_\alpha(A, \varepsilon)$$

There exists unique $d \geq 0$ such that

$$H_\alpha(A) = \begin{cases} 0, & \alpha > d \\ \infty, & \alpha < d \end{cases}$$

$\rightarrow$ d is the Hausdorff dimension

$\rightarrow$ each countable set has Hausdorff dimension 0.

proof:

$\forall \alpha \geq 0$ let $\gamma(\varepsilon) = \{B_i : B_i = B_{\frac{\varepsilon}{2^i}}(x_i)\}$, where x_i is a sequence satisfying $x_i \in A$ and $\bigcup_{i=1}^{\infty} x_i = A$. Then

$$H_\alpha(A, \varepsilon) \leq \sum_{i=1}^{\infty} \frac{\varepsilon^\alpha}{(2^i)^\alpha} = \sum_{i=1}^{\infty} \frac{\varepsilon^\alpha}{(2^\alpha)^i} \leq C\, \varepsilon^\alpha.$$

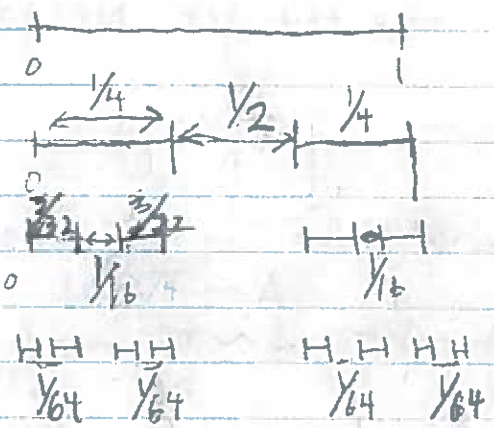
$$\Rightarrow \lim_{\varepsilon \to \infty} H_\alpha(A, \varepsilon) = H_\alpha(A) = 0.$$

Example (Fat Fractal)

Does every fractal have measure 0?

$3/16$ No.

$\frac{3}{2^{i+2}}$ $\frac{1}{4} - \frac{1}{16}$



S_0

$S_1 \rightarrow L = 1/2$

$S_2 \rightarrow L = 4 \cdot 3/32 = 3/8$

$S_3 \rightarrow L = 8 \cdot 21/8 \cdot 32 \cdot 2 = 21/64$

$\frac{2^{i-1}}{2^{2i}}$

Remove $1/2^n$ from each subinterval

$\frac{3}{32} \cdot \frac{1}{8}$

$\frac{3}{32} - \frac{2 \cdot 1}{32 \cdot 8}$

$21/32$

$$C_{Fat} = \bigcap_{i=1}^{\infty} S_i$$

Box Dimension:

Let $\varepsilon_n = 3/2^{n+2}$, this corresponds to $N = 2^n$.

The box dimension is then

$$d = \lim_{n \to \infty} \frac{\ln(2^n)}{\ln(2^{n+2}/3)} = 1.$$

The width removed is $\sum_{i=1}^{\infty} \frac{1}{4^i} \cdot 2^i = \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2}$

The fat fractal has length ½!

Strange repellor

$$x_{n+1} = \begin{cases} 3x, & x < 1/2 \\ 3x - 2, & x > 1/2 \end{cases}$$