Chapter 3: Bifurcations

Systems with parameters can have drastic qualitative changes as a parameter is varied.

Examples:

* Euler Buckling, * Turbulence * outbreaks of epidemics.
* catastrophic environmental collapse

Framework:

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}, \ \mu \in \mathbb{R} \text{ parameter}$$

Study behaviour of fixed points under parameter changes

Suppose $x^*(\mu)$ is a fixed point.

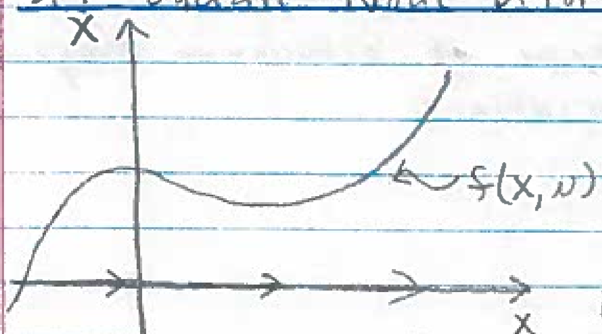
1. $\left.\frac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} > 0 \rightarrow$ unstable fixed point

2. $\left.\frac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} < 0 \rightarrow$ stable fixed point

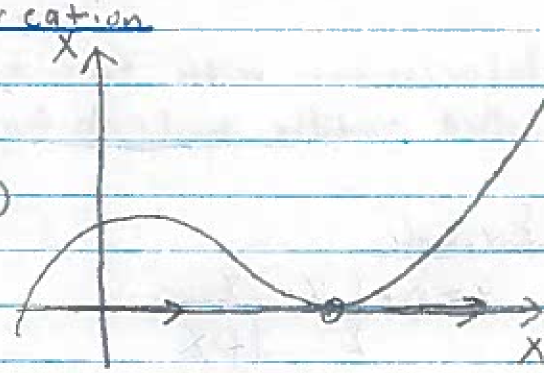
3. $\left.\frac{\partial f}{\partial x}\right|_{(x^*(\mu), \mu)} = 0 \rightarrow$ possibly semistable

We can think of x^* as a (multivalued) function of μ.

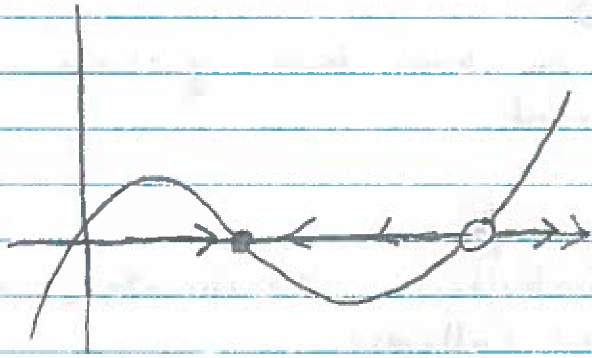
3.1 Saddle-Node Bifurcation



$\mu < \mu^*$



$\mu = \mu^*$

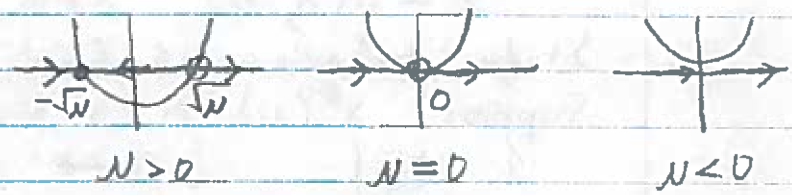


$\mu = \mu^*$

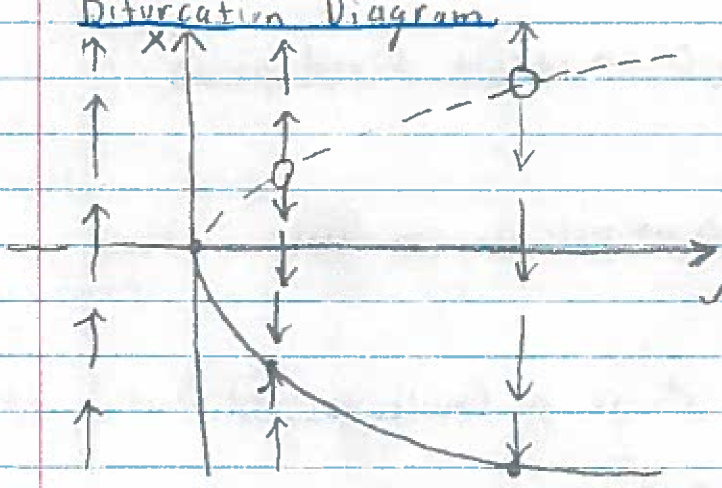
The bifurcation point is the value of μ where the fixed point changes stability.

Prototypical Example:

$$\dot{x} = -\mu + x^2$$



Bifurcation Diagram



location of fixed points as a function of μ. The phase portraits can be reconstructed

* The bifurcation diagram does not typically contain the arrows

Bifurcations with this type of bifurcation diagram are called saddle node bifurcations.

Example

$$\dot{x} = r + \frac{1}{2}x - \frac{x}{1+x}$$

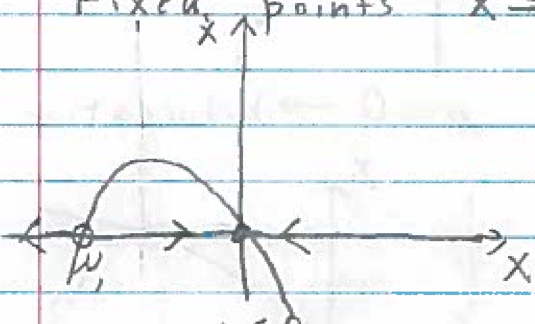
3.2 Transcritical Bifurcation

Many systems have a fixed point that cannot change position (population $P=0$, for example). However, the stability of the fixed point can change.

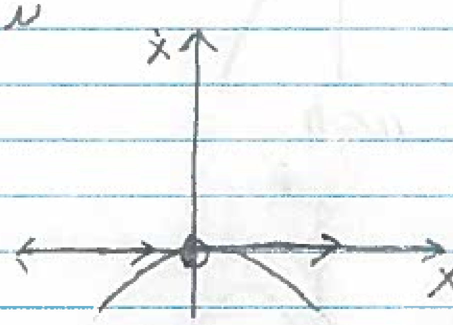
Typical Example:

$$\dot{x} = \mu x - x^2$$

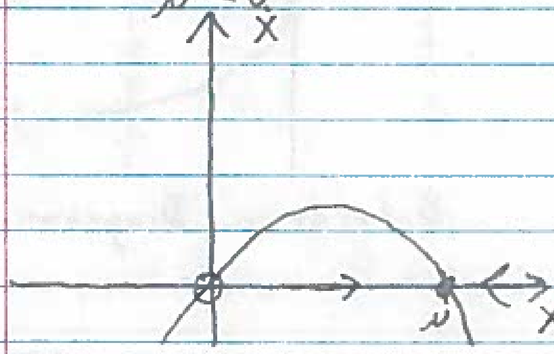
Fixed points $x = 0, \mu$



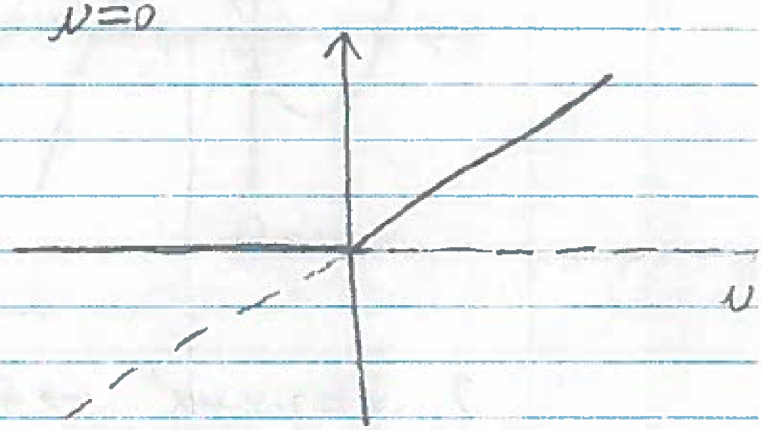
$\mu < 0$



$\mu = 0$



$\mu = 0$

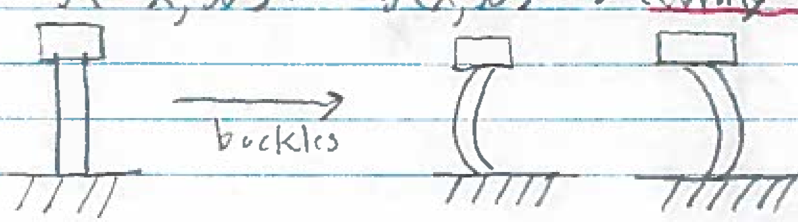


Bifurcation diagram.

3.4 Pitchfork Bifurcation

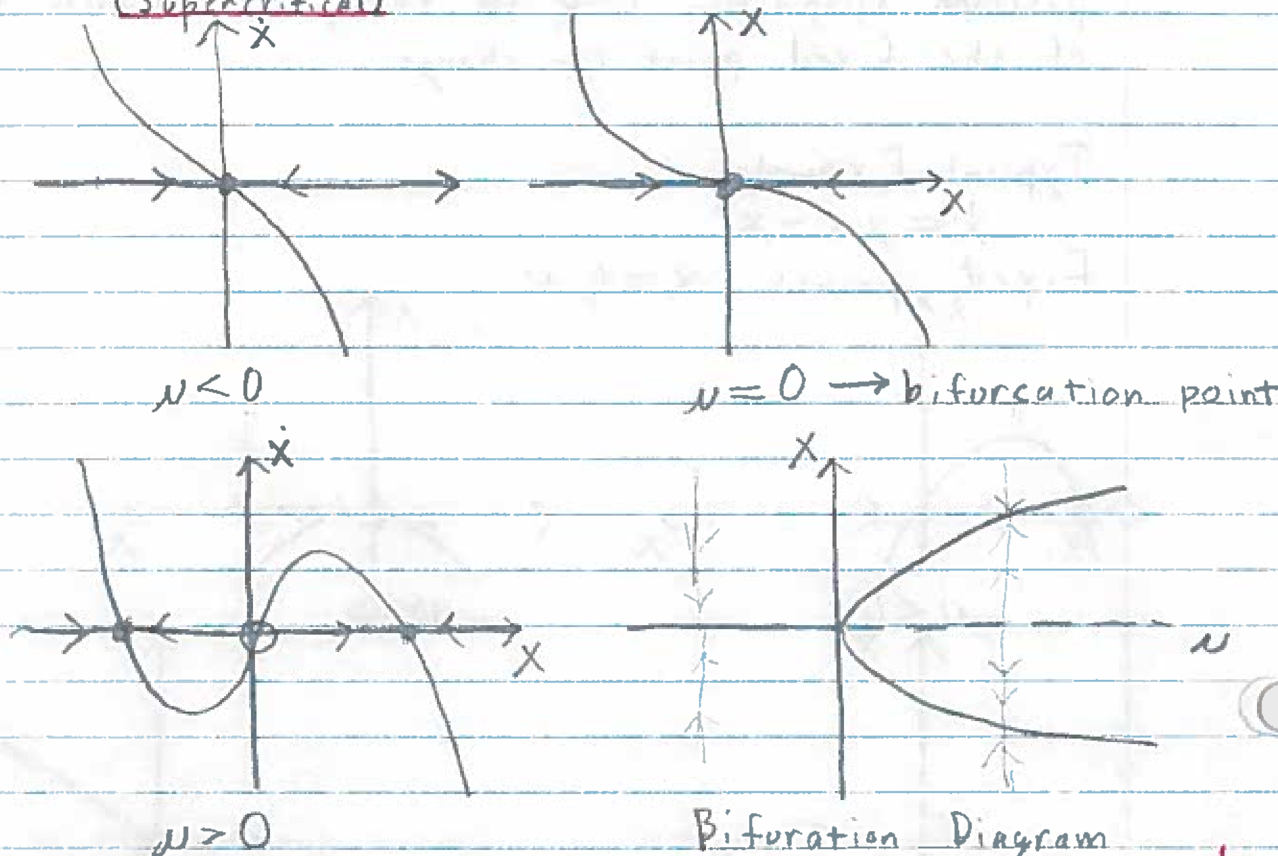
Many physical systems have odd symmetry:

$$f(-x, \mu) = -f(x, \mu) \Rightarrow \text{(evenly symmetric potential energy)}$$



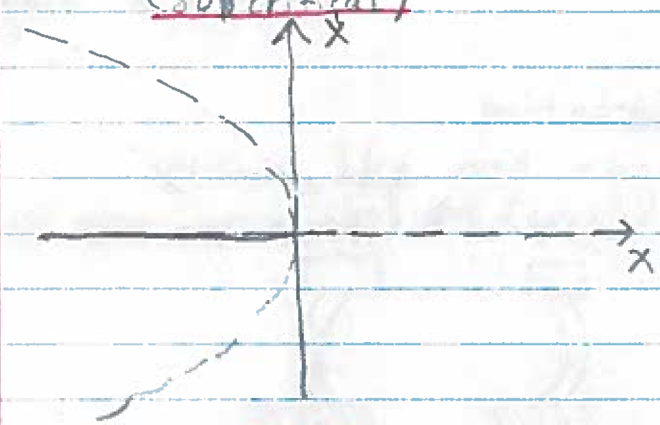
Prototypical Examples:

1. $\dot{x} = \mu x - x^3 \rightarrow$ fixed points: $x^* = 0, \pm\sqrt{\mu}$
(supercritical)



L4

2. $\dot{x} = \mu x + x^3 \rightarrow$ fixed points: $x^* = 0, \pm\sqrt{-\mu}$
(subcritical)

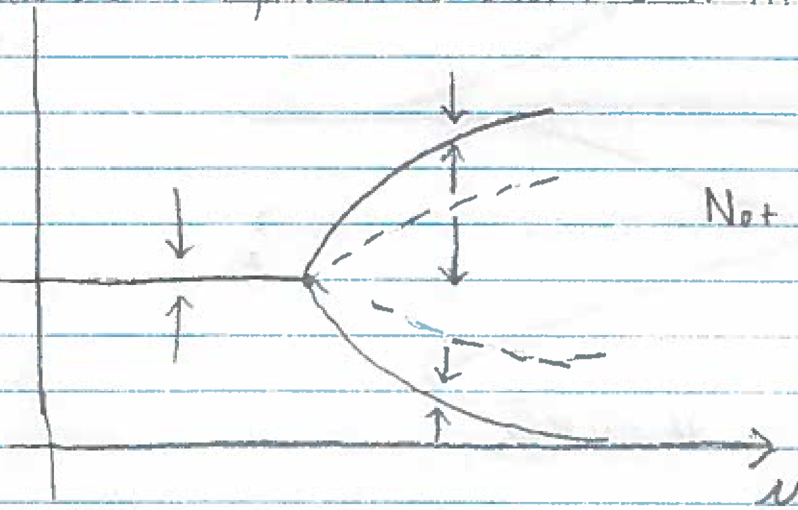


The fixed point switches to unstable in fact, the solution goes to ∞ in finite time

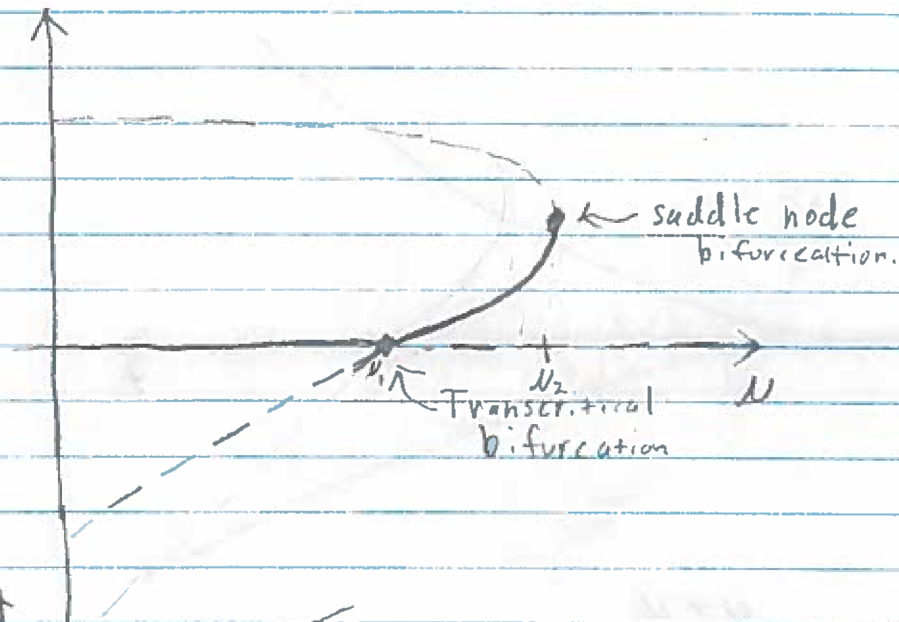
L4

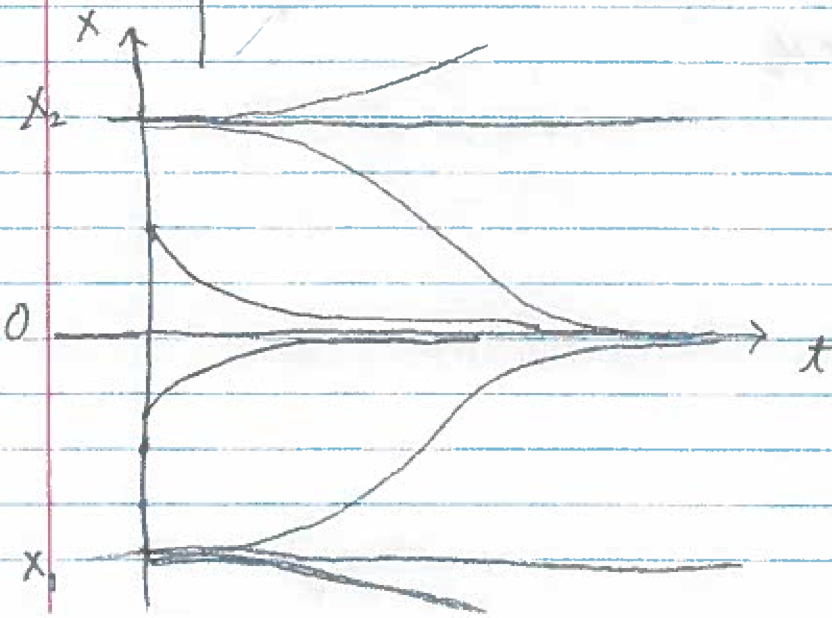
Reverse Process

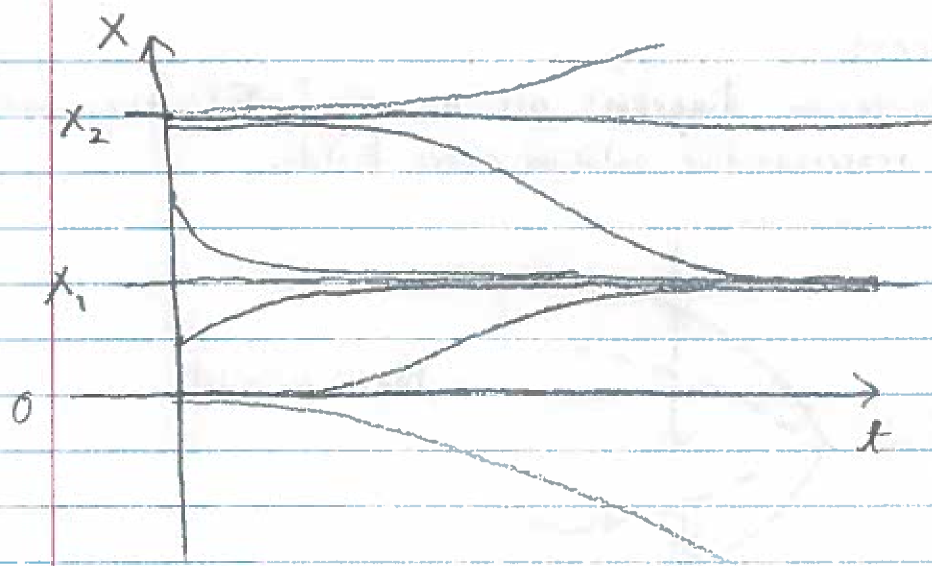
Which bifurcation diagrams are possible? If they are possible sketch several representitive solution curve fields.



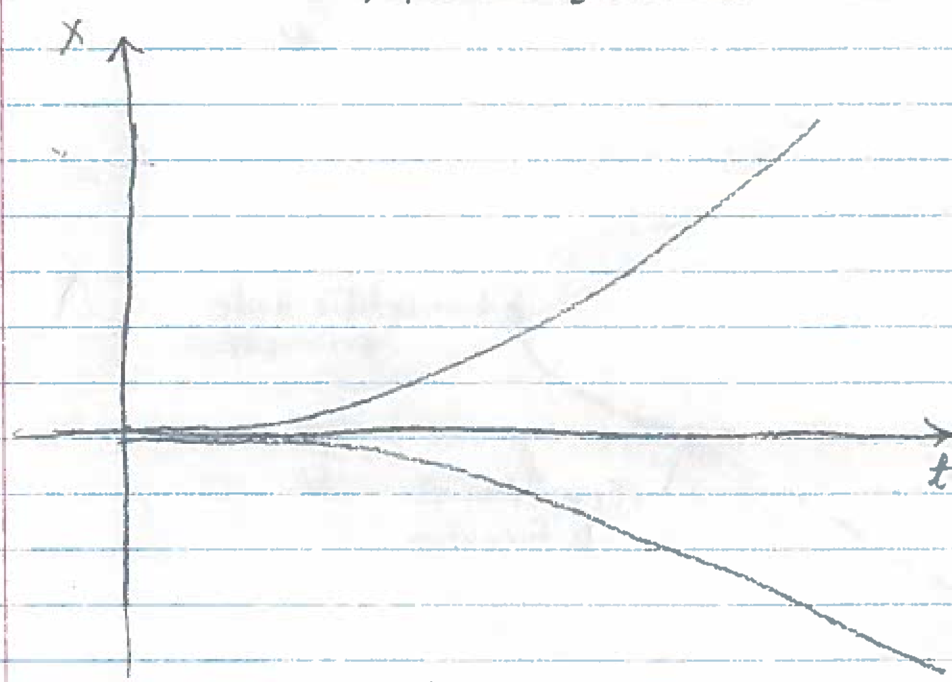
Not possible







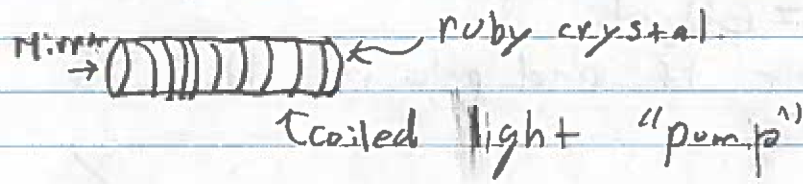
$\mu_1 < \mu < \mu_2$



$\mu > \mu_2$

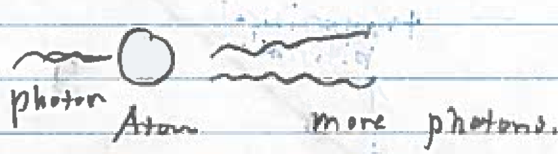
example:

Model of a laser.



$n \sim$ number of photons

$N \sim$ number of excited atoms.



$$\dot{n} = GN\cdot n - Kn$$

$[G] = \frac{1}{\text{time}}$, Gain coefficient.

$[K] = \frac{1}{\text{time}}$ ~ typical lifetime of photon.

$$N(t) = N_0 - \alpha n$$

$[\alpha]$ - rate that atom drops back to rest state.

N_0 - number of excited atoms with no photons

$$\Rightarrow \dot{n} = G(N_0 - \alpha n)n - Kn$$
$$= (GN_0 - K)n - \alpha G n^2$$

Fixed points:

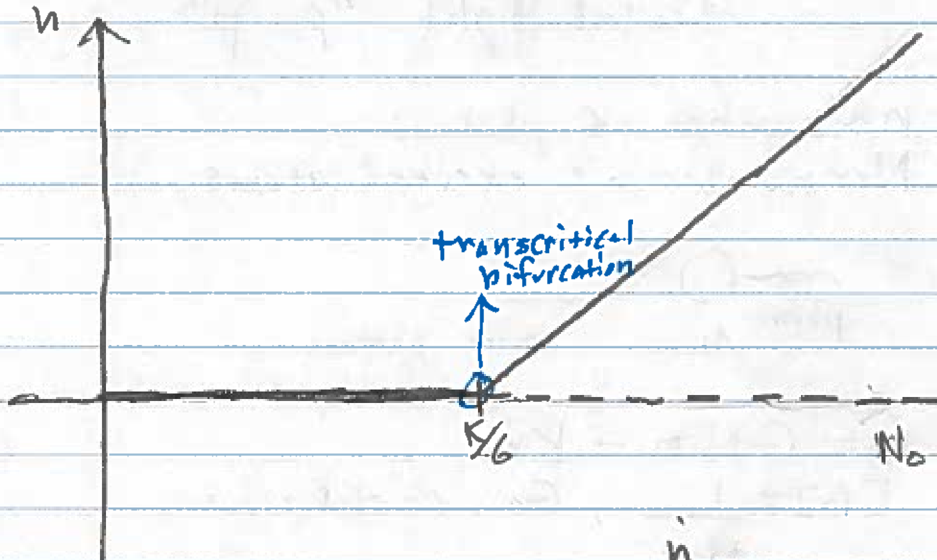
$$0 = n[\, GN_0 - K \;\; - \alpha G n]$$
$$\Rightarrow n = 0,\; n = \frac{GN_0 - K}{G\alpha}$$

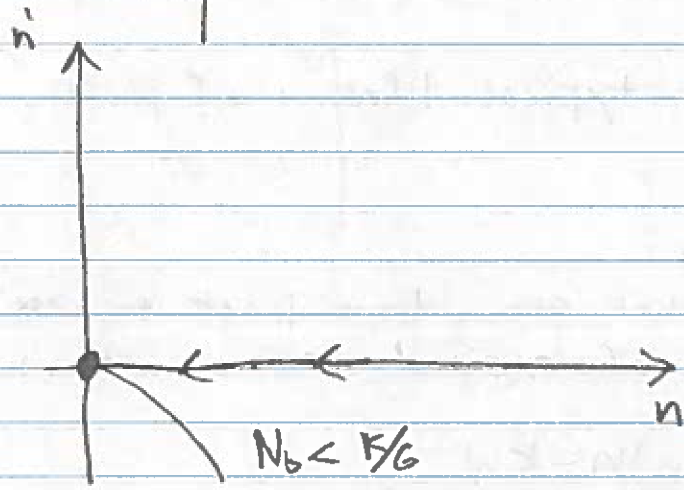
Let $f(n) = (GN_0 - K)n - \alpha n^2$

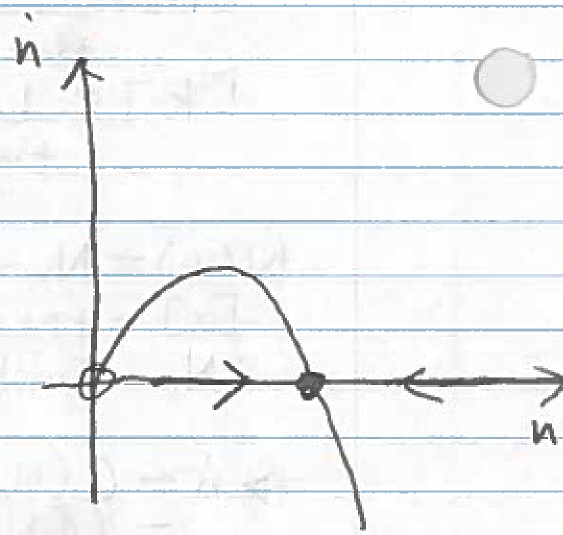
$\Rightarrow f'(n) = GN_0 - K - 2\alpha n$

$\Rightarrow f'(0) = GN_0 - K$

0 is stable if and only if $N_0 < K/G$.







example:

Model of lake:

$$\dot{N} = rN(1 - N/k) - H$$

logistic growth. Harvesting

$\tau = rt$, $x = N/K$ → dimensionless scales

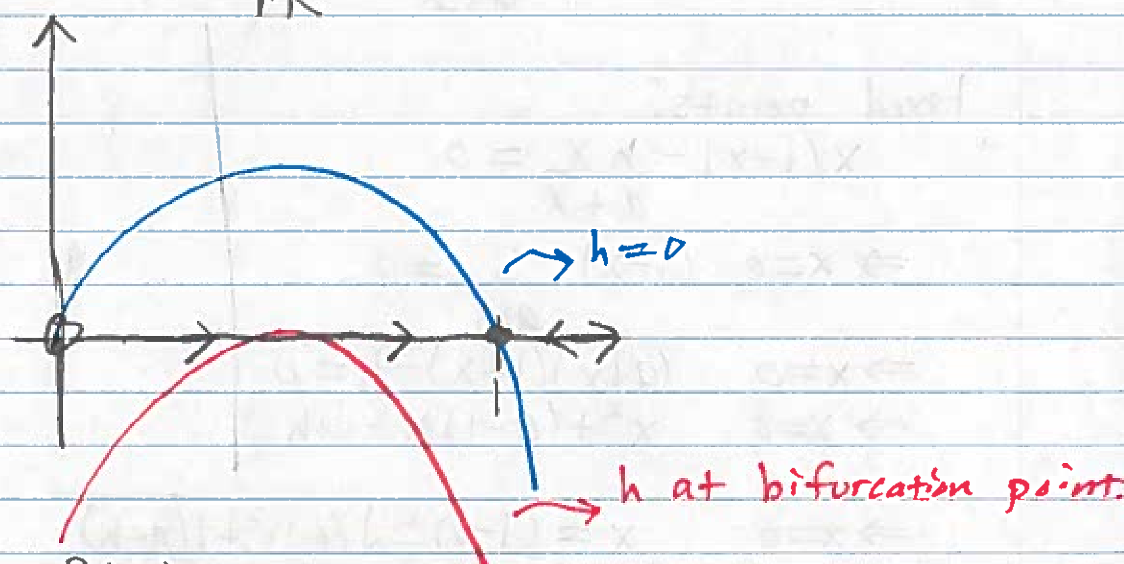
$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = r\frac{d}{d\tau}$$

$$\Rightarrow \frac{dN}{dt} = r\frac{dN}{d\tau} = r\frac{d}{d\tau}(Kx) = rK\frac{dx}{d\tau}$$

$$\Rightarrow rK\frac{dx}{d\tau} = r \cdot Kx(1-x) - H$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - h,$$

where $h = \frac{H}{rK}$.

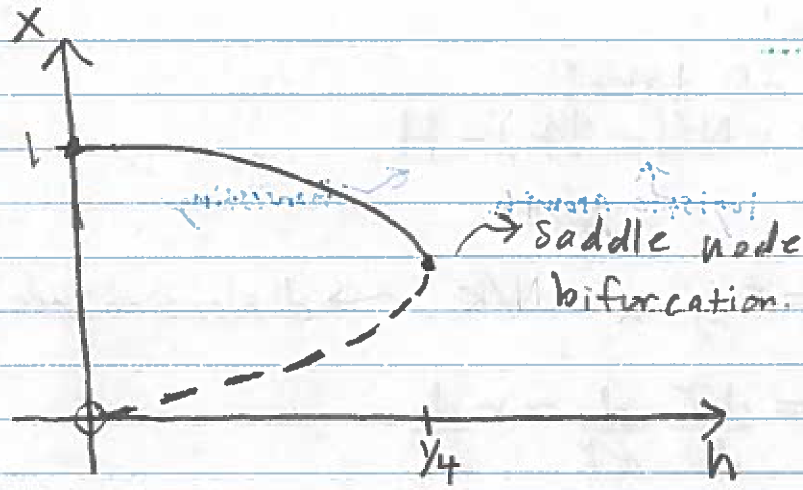


Let $f(x) = x(1-x) - h$

$f'(x) = 1 - 2x = 0$

$\Rightarrow x = 1/2$

Now $f(1/2) = 1/4 - h = 0 \Rightarrow h = 1/4$ is the bifurcation point.



example:

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\cdot\frac{N}{A+N}$$

Survivability

Let $x = N/K$, $\tau = rt$.

$$\Rightarrow \dot{x} = x(1-x) - h\frac{x}{a+x}.$$

Fixed points:

$$x(1-x) - \frac{hx}{a+x} = 0$$

$$\Rightarrow x=0, \quad (1-x) - \frac{h}{a+x} = 0$$

$$\Rightarrow x=0, \quad (a+x)(1-x) - h = 0$$

$$\Rightarrow x=0, \quad x^2 + (a-1)x - a + h$$

$$\Rightarrow x=0, \quad x = \frac{(1-a) \pm \sqrt{(a-1)^2 + 4(a-h)}}{2}$$

When $h=0$

$$x = \frac{(1-a) \pm \sqrt{(a-1)^2+4a}}{2}$$

$$= \frac{(1-a) \pm 1 + a}{2} = 1, -a$$

Fix a, what happens as h changes?

Look at stability at $x=0$. Let $f(x) = x(1-x) - \frac{hx}{a+x}$

then

$$f'(x) = 1 - 2x - \frac{[(a+x)\cdot h - hx]}{(a+x)^2}$$

$$\Rightarrow f'(0) = 1 - \frac{ah}{a^2} = \frac{a-h}{a}.$$

* 0 is stable if $h > a$,
unstable if $h < a$

Other roots exist if $(a-1)^2 + 4(a-h) > 0$

$$a - h > -\frac{(a-1)^2}{4}$$

$$\Rightarrow h < \frac{(a-1)^2}{4} + a.$$

Case 1:

$$h < a < \frac{(a-1)^2}{4} + a$$

Three fixed points.

→ Robust survival of fish.

Case 2:

$$a < h < \frac{(a-1)^2}{4} + a$$

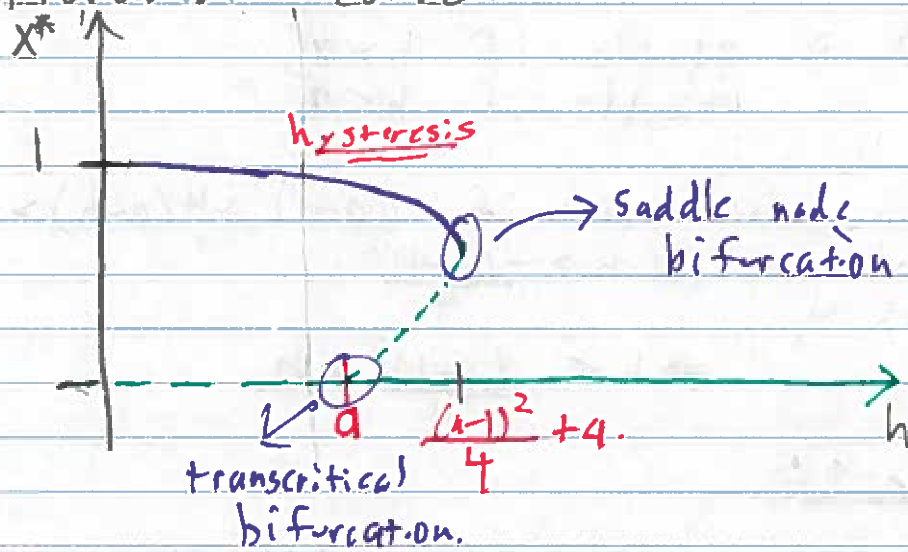
Three fixed points. → Species endangered.

Case 3:

$\frac{(a-1)^2}{4} + a < h$

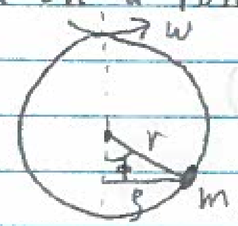
only one fixed point.

Bifurcation curve:



Example:

Bead on a rotating hoop



$$\underbrace{mr\ddot{\phi}}_{\text{net force}} = \underbrace{-b\dot{\phi}}_{\text{friction}} \underbrace{- mg\sin(\phi)}_{\text{gravity}} + \underbrace{mrw^2\sin(\phi)\cos(\phi)}_{\text{centrifugal}} = 0$$

$[m] = M$

$[r] = L$

$[b] = T^{-1}LM$

$[g] = LT^{-2}$

$[w] = T^{-1}$

Rescale:

$\tau = T_{sc}\, t$

We will choose T_{sc} later.

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau}$$

$$\Rightarrow mr\,T_{sc}^2 \frac{d^2\phi}{d\tau^2} = -b\,T_{sc}\frac{d\phi}{d\tau} - mg\sin(\phi) + mrw^2\sin(\phi)\cos(\phi) = 0.$$

Divide by mg to nondimensionalize:

$$\frac{r}{g}T_{sc}^2\frac{d^2\phi}{d\tau^2} = -\frac{b}{mg}T_{sc}\frac{d\phi}{d\tau} - \sin(\phi) + \frac{rw^2}{g}\sin(\phi)\cos(\phi) = 0.$$

In order to reduce to a first order system we need

$$\frac{b}{mg}T_{sc} = O(1) \quad \text{and} \quad \frac{r}{g}T_{sc}^2 \ll 1$$

$$\Rightarrow \boxed{T_{sc} = \frac{mg}{b} \quad \Rightarrow \varepsilon = \frac{rm^2g}{b^2} \ll 1}$$

$$\Rightarrow \varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin(\phi) + \gamma\sin(\phi)\cos(\phi)$$

$$\boxed{\gamma = \frac{rw^2}{g}}$$

The 1-D system is then

$$\frac{d\phi}{d\tau} = -\sin(\phi) + \gamma\sin(\phi)\cos(\phi) = -\sin(\phi) + \frac{\gamma}{2}\sin(2\phi)$$

$$\frac{d\phi}{d\tau} = \sin(\phi)\gamma\left(\cos(\phi) - \frac{1}{\gamma}\right)$$

Fixed points:

$\phi = 0, \pi$, $\cos(\phi) = \frac{1}{\gamma}$

$$\left.\frac{df}{d\phi}\right|_0 = -1 + \gamma$$

$\Rightarrow$ $\gamma > 1$, $\phi = 0$ is unstable

$\gamma < 1$, $\phi = 0$ is stable

$$\left.\frac{df}{d\phi}\right|_\pi = 1 + \gamma$$

$\Rightarrow$ $\phi = \pi$ is unstable.

$\cos(\phi) = \frac{1}{\gamma}$