Chapter 5 and 6: Phase Plane

Differential Equations with two variables (x,y):

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

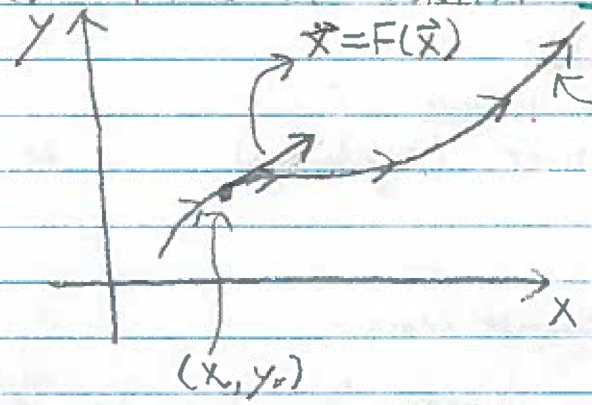
where $(x,y) \in \mathbb{R}^2$ and $f,g : \mathbb{R}^2 \to \mathbb{R}$ are continuously different

We can also write this system in the form:

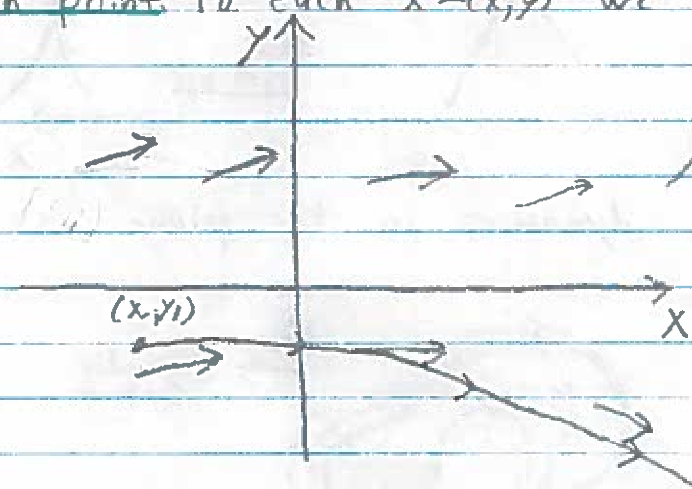
$$\dot{\vec{x}} = F(x,y)$$

where $F : \mathbb{R}^2 \to \mathbb{R}^2$.

We can think of solution curves as flows.



We can think of $F(\vec{x})$ as assigning a velocity vector to each point: to each $\vec{x} = (x,y)$ we assign the vector $F(\vec{x})$



Particular Solutions of interest:

1. Fixed Points: Each $\vec{x}_0$ satisfy $F(\vec{x}_0) = 0$
2. Periodic Trajectories: $\vec{x}(t)$ is periodic if $\exists T > 0$ such that $\vec{x}(t+T) = \vec{x}(t)$ $\forall t$ and $\vec{x}(t)$ is

not a fixed point.

Theorem - Assume $F: \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable for all $\vec{x} \in \mathbb{R}^n$, then for each $\vec{x}_0 \in \mathbb{R}^n$ the system

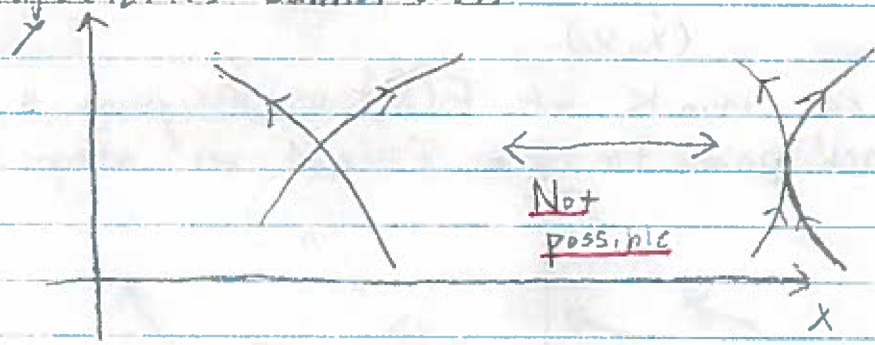
$$\dot{\vec{x}} = F(\vec{x})$$
$$\vec{x}(0) = \vec{x}_0$$

1 has a solution $\vec{x}(t)$ in the interval $(-\tau, \tau)$ for some $\tau > 0$ and the solution is 2 unique. The map $\vec{x}_0 \mapsto \vec{x}(t, \vec{x}_0)$ is 3 continuously differentiable for each fixed t.

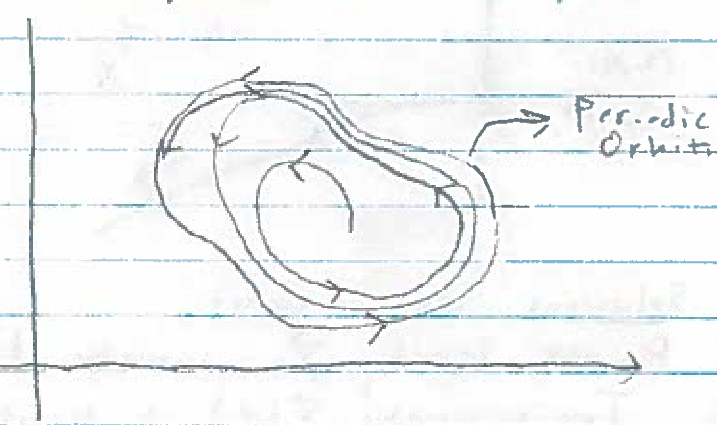
$\Rightarrow$ 1. Solutions exist.
2. Solutions are unique
3. Flows are nice (regularity)

Consequences:

1. Trajectories cannot cross



2. Constrains dynamics in the plane



Linear Systems:

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

We can write

$$\dot{\vec{x}} = A\vec{x}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. If $\vec{x}_1, \vec{x}_2$ satisfy $\dot{\vec{x}} = A\vec{x}$ then so does any linear combination
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2.$$
Therefore, it suffices to find two solutions $\vec{x}_1, \vec{x}_2$ with $\vec{x}_1(0)$ and $\vec{x}_2(0)$ linearly independant to generate all solutions.

2. If λ is an eignvalue of A with corresponding eigenvector $\vec{v}$ then $A\vec{v} = \lambda\vec{v}$. Therefore,
$$\vec{x} = e^{\lambda t}\vec{v}$$
is a solution:

proof:
$$\frac{d\vec{x}}{dt} = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}Av = A\vec{x}$$

3. All solutions can be written in the form:
$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2,$$
where λ_1, λ_2 and $\vec{v}_1, \vec{v}_2$ are the eigenvalues and corresponding eigenvectors, respectively.

4. The eigen directions tell us where the flow is invariant (Invariant manifolds)

5. At $t=0$ we have
$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$$

Examples:

1. $\dot{\vec{x}} = \begin{pmatrix}1 & 3\\ 0 & -2\end{pmatrix}\vec{x}$

$= A\vec{x}$

$\lambda_1 \cdot \lambda_2 = \det(A) = -2$

$\lambda_1 + \lambda_2 = \mathrm{tr}(A) = -1$

$\Rightarrow \lambda_2 = -1 - \lambda_1$

$\Rightarrow \lambda_1(1+\lambda_1) = 2$

$\lambda_1^2 + \lambda_1 - 2 = 0$

$$\boxed{\begin{matrix}\lambda_1 = -2\\ \lambda_2 = 1\end{matrix}}$$

Clearly,

$\vec{v}_2 = \begin{pmatrix}1\\0\end{pmatrix}$

To determine $\vec{v}_1$:

$(A + 2I)\vec{v}_2 = 0$

$\Rightarrow \begin{pmatrix}3 & 3\\ 0 & 0\end{pmatrix}\vec{v}_2 = 0$

$\Rightarrow \vec{v}_1 = \begin{pmatrix}1\\-1\end{pmatrix}$

The general solution is then:

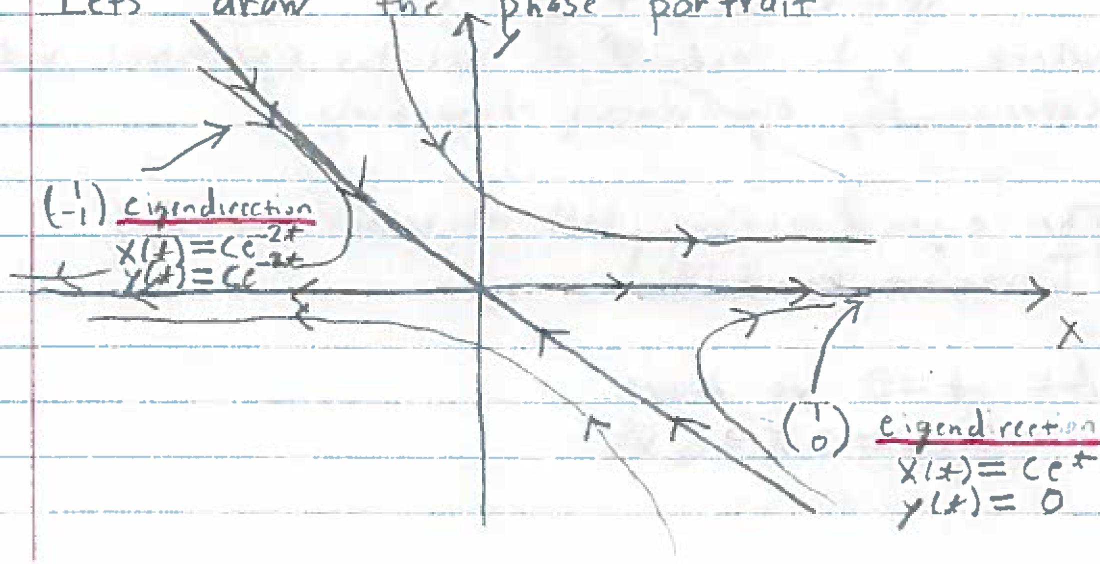
$\vec{x} = c_1 e^{t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$

$= \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$

As $t \to \infty$ $x(t) \to \pm\infty$

$t \to \infty$ $y(t) \to 0$

Lets draw the phase portrait



2. $\dot{\vec{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$

$= A\vec{x}$

Eigenvalues satisfy

$\lambda_1 \cdot \lambda_2 = 1$

$\lambda_1 + \lambda_2 = 0$

$\Rightarrow \lambda_1 = i$

$\lambda_2 = -i$

The eigenvectors are

$\vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

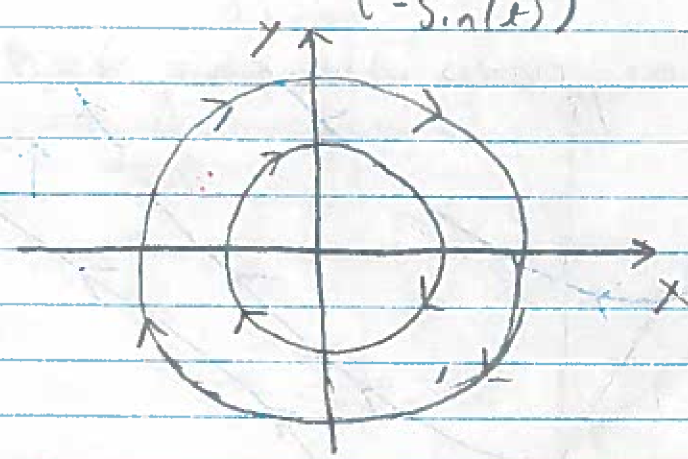
The general solution is:

$\vec{x}(t) = c_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

We of course want real valued solutions so we set

$\vec{x}(t) = c_1 \, \mathrm{Re}\left[e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} \right] + c_2 \, \mathrm{Im}\left[e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} \right]$

$\Rightarrow \vec{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$



Periodic Orbit

Period 2π

Example: (Richardson's Arm's Race)

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy.$$

$x \rightarrow$ expendature on arms by nation x

$y \rightarrow$ expendature on arms by nation y

$a, d > 0$ nation x or y are arms dealers.

$a, d < 0$ nation x or y have limited resourses

$b, d > 0$ nation x or y responds aggresively to other nation.

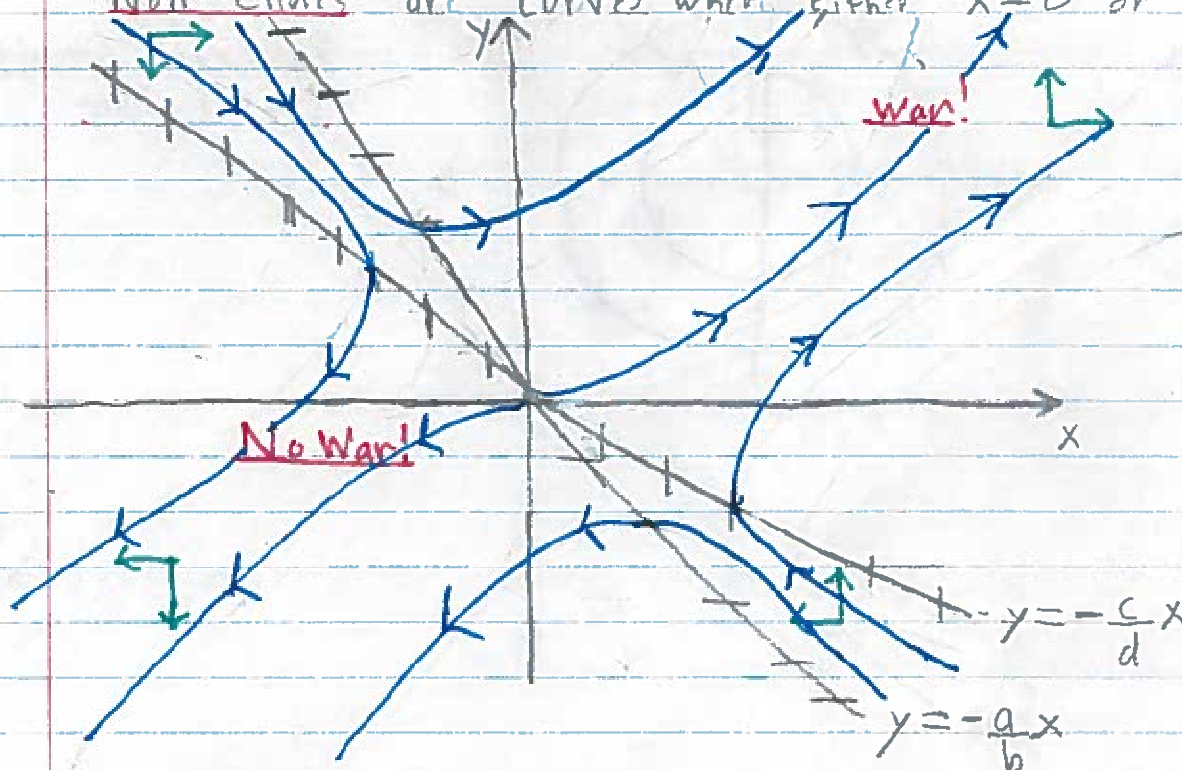
$b, d < 0$ nation x or y reduces spending depending on the other nations arms.

Case 1:

$a, d > 0$

$b, d > 0$

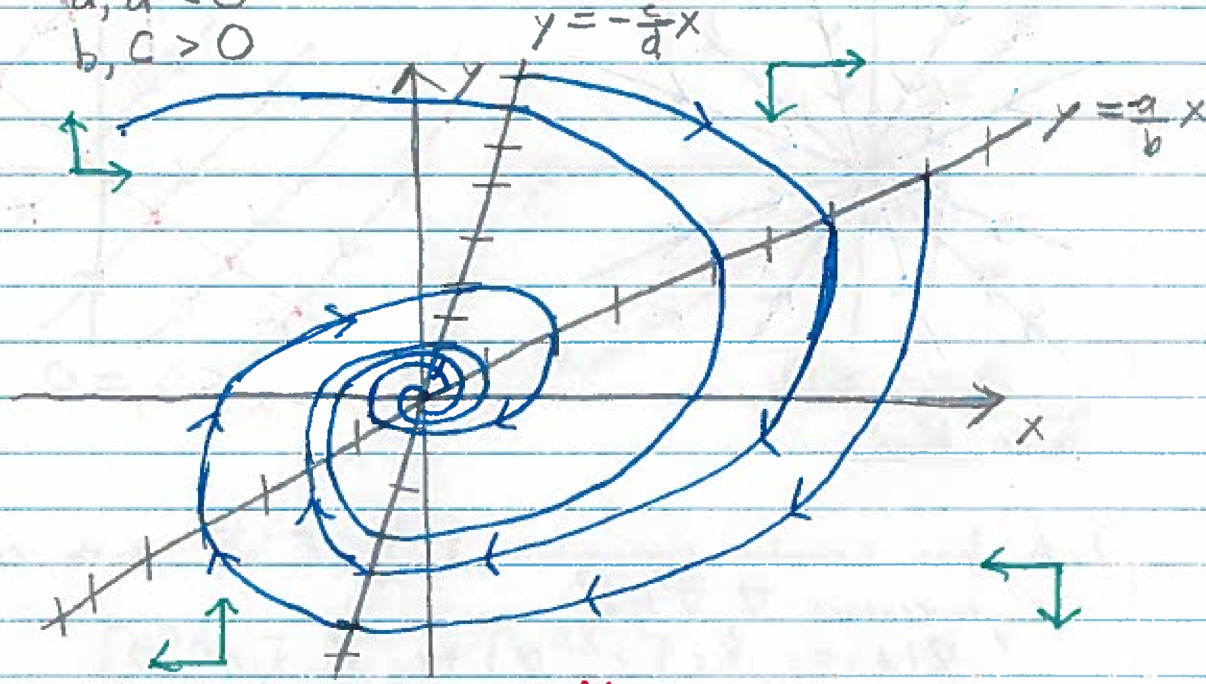
Null-clines are curves where either $\dot{x} = 0$ or $\dot{y} = 0$



Case 2:

$a, d < 0$

$b, c > 0$



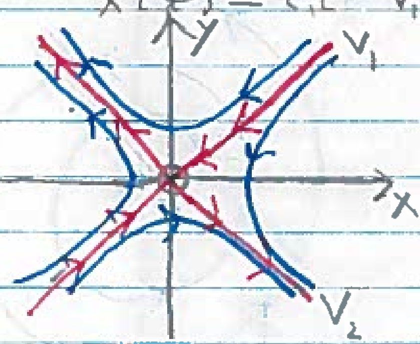
World Peace!!

Everybody focuses on Internal Nation Building.

Eigenvalue analysis:

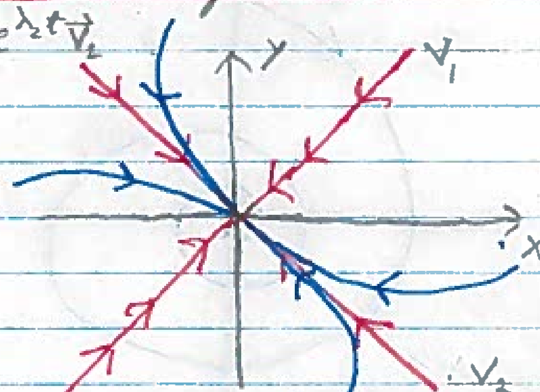
1. $\dot{\vec{x}} = A\vec{x}$, A has two real eigenvalues

$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$



$\lambda_1 < 0 < \lambda_2$

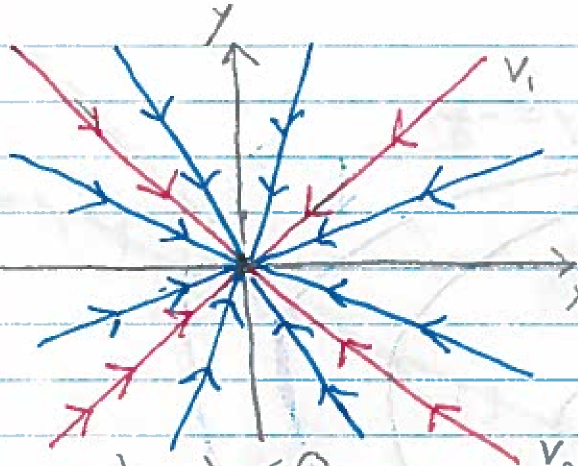
Saddle node



$\lambda_1 < \lambda_2 < 0$

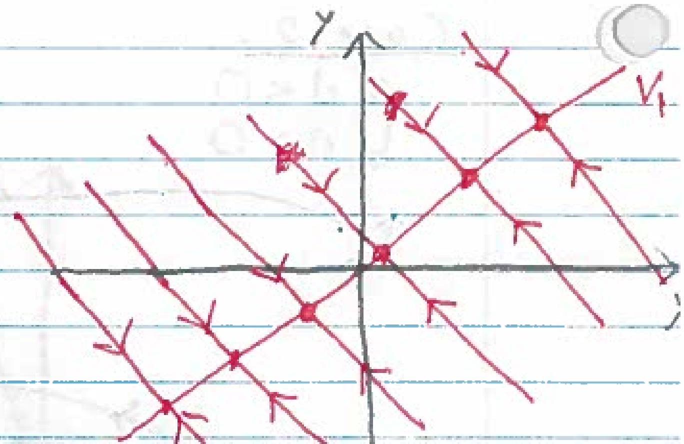
Stable node

(V_1 is the fast direction)



$\lambda_1 = \lambda_2 < 0$

Star Node



$\lambda_2 < \lambda_1 = 0$

2. A has complex eigenvalues $\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ with complex eigenvectors $\vec{v}, \bar{\vec{v}} \in \mathbb{C}^2$.

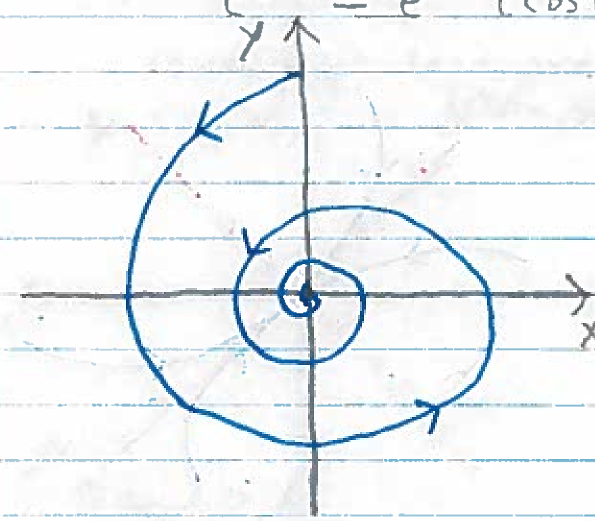
$$\vec{x}(t) = c_1 \operatorname{Re}[e^{\lambda t}\vec{v}] + c_2 \operatorname{Im}[e^{\lambda t}\vec{v}]$$

If we write

$$\lambda = \nu + i\mu$$

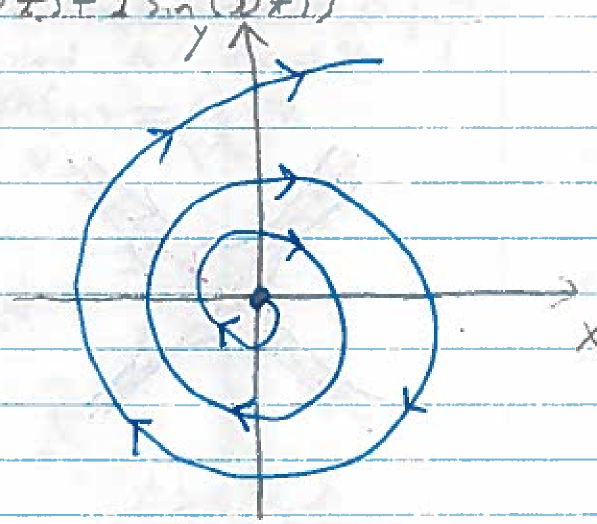
then

$$e^{\lambda t} = e^{\nu t}(\cos(\mu t) + i\sin(\mu t))$$



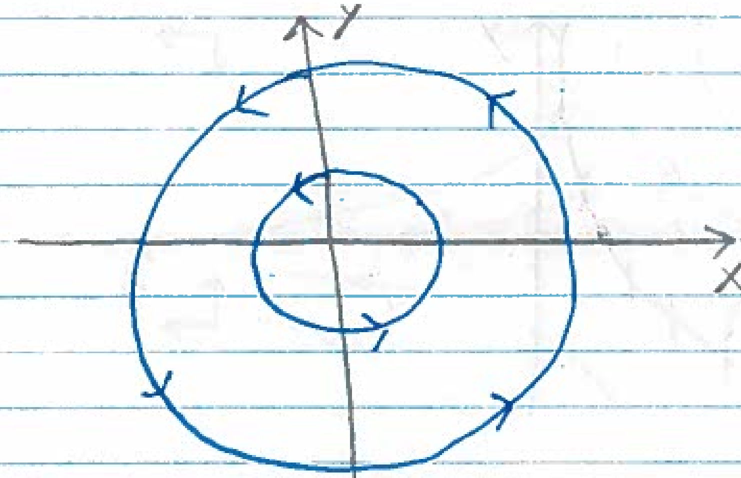
$\operatorname{Re}(\lambda) < 0$

(Stable Spiral)



$\operatorname{Re}(\lambda) > 0$

(Unstable Spiral).



$Re(\lambda) = 0$
(center)

Notation:

1. A is called hyperbolic if $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.
2. We say that the fixed point $x=0$ of $\dot{x} = Ax$ is
   a.) an attractor if $\vec{x}(t) \to 0$ as $t \to \infty \Rightarrow Re(\lambda_1, \lambda_2) < 0$
   b.) a repellor if $\vec{x}(t) \to 0$ as $t \to -\infty \Rightarrow Re(\lambda_1, \lambda_2) > 0$
   c.) a saddle if $\lambda_1 < 0 < \lambda_2$
   d.) nonhyperbolic if $Re(\lambda_1) = 0$ or $Re(\lambda_2) = 0$

Example:

Sketch the phase portrait of

$$\begin{cases} \dot{x} = x + e^{-y} = f(x,y) \\ \dot{y} = -y \quad = g(x,y). \end{cases}$$

Fixed points: $y = 0$ and $x = -1$.

Nullclines:

a.) $-x = e^{-y} \Rightarrow y = -\ln(-x)$ N1: $\frac{dx}{dt} = 0$

b.) $y = 0$ N2: $\frac{dy}{dt} = 0$