

Chapter 7: Limit Cycles

Definition - A closed trajectory is a limit cycle if it is separated from all other closed trajectories.

a.) A limit cycle is stable if there is a tubular neighborhood such that trajectories that enter the neighborhood approach the limit cycle as $t \rightarrow \infty$.

b.) A limit cycle is unstable if it is not stable.

Example:

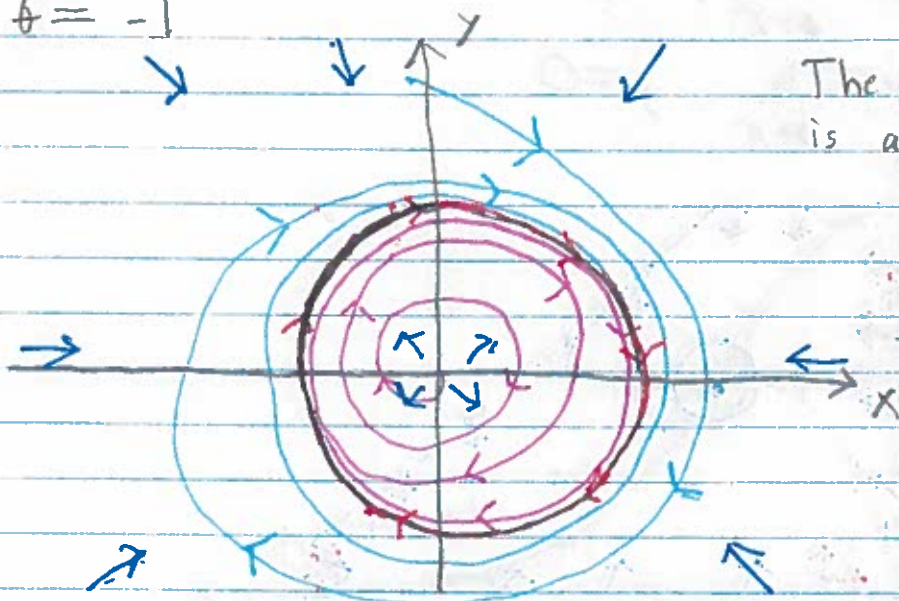
$$\begin{aligned} \dot{x} &= -y + x(1 - \sqrt{x^2 + y^2}) & x &= r \cos \theta \\ \dot{y} &= x + y(1 - \sqrt{x^2 + y^2}) & y &= r \sin \theta \end{aligned}$$

$$\begin{aligned} \dot{x} &= \dot{r} \cos \theta - r \sin \theta \dot{\theta} \\ \dot{y} &= \dot{r} \sin \theta + r \cos \theta \dot{\theta} \end{aligned}$$

$$\begin{bmatrix} \dot{r} \\ \dot{\theta} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \dot{r} &= \cos \theta (-r \sin \theta + r \cos \theta (1-r)) + \sin \theta (r \cos \theta + r \sin \theta (1-r)) \\ \dot{\theta} &= \frac{-\sin \theta}{r} (-r \sin \theta + r \cos \theta (1-r)) + \frac{\cos \theta}{r} (r \cos \theta + r \sin \theta (1-r)) \end{aligned}$$

$$\begin{aligned} \Rightarrow \dot{r} &= r(1-r) \\ \dot{\theta} &= -1 \end{aligned}$$



The curve $x^2 + y^2 = 1$ is a stable limit cycle.

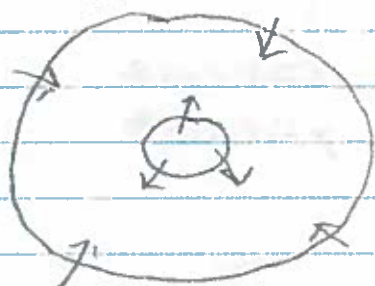
Poincaré-Bendixon Theorem - Consider $\dot{\vec{x}} = F(\vec{x})$, with F continuously differentiable. Assume $R \subset \mathbb{R}^2$ is closed and bounded.

(i) R does not contain any fixed points.

(ii) There exists $\vec{x}(0) \in R$ so that $\vec{x} \in R$ for all $t \geq 0$.

Then R contains a limit cycle.

Typical Application:



$R =$ trapping region.

Example:

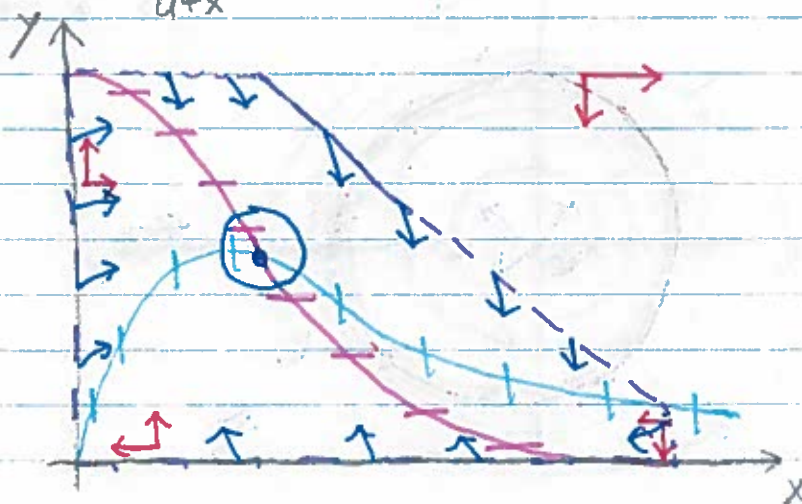
$$\begin{aligned}\dot{x} &= -x + ay + x^2y \\ \dot{y} &= b - ay - x^2y\end{aligned}$$

Global Analysis:

Null clines:

$$y = \frac{x}{a+x^2}, \quad \dot{x} = 0$$

$$y = \frac{b}{a+x^2}, \quad \dot{y} = 0$$



When is $\frac{dy}{dx} < -1$

$$\Rightarrow b - ay - x^2 y < -x - ay - x^2 y$$

$$\Rightarrow b < x$$

Therefore, if $x > b$ we know $\frac{dy}{dx} < -1$.

The fixed point is at $x = b, y = \frac{b}{a+b^2}$.

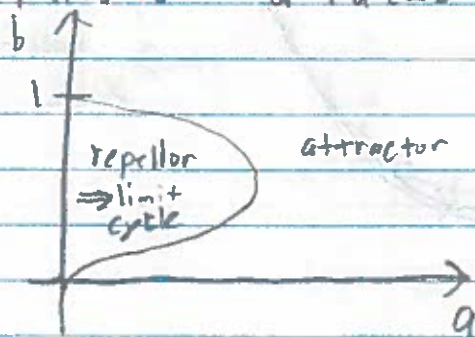
$$J = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix}$$

$$J|_{(x,y)} = \begin{pmatrix} -1 + \frac{2b}{a+b^2} & a + b^2 \\ -\frac{2b}{a+b^2} & -a - b^2 \end{pmatrix} = A$$

$$\det(A) = a + b^2 > 0$$

$$\text{Tr}(A) = -1 - a - b^2 + \frac{2b^2}{a+b^2}$$

$$\text{Tr}(A) = 0 \Leftrightarrow a^2 + a(2b^2 + 1) + b^2(b^2 - 1) = 0$$



Van der Pol Oscillator

$$\ddot{x} + \frac{1}{\epsilon}(x^2 - 1)\dot{x} + x = 0$$

* $\frac{1}{\epsilon}x^2 - 1$ is like a damping or pumping term

- If $x^2 < 1$ energy is pumped in (forcing)
- If $x^2 > 1$ energy is pumped out (friction)

In fact:

$$\frac{d}{dt} E = \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{x^2}{2} \right) = -\frac{1}{\epsilon} (x^2 - 1) \dot{x}^2$$

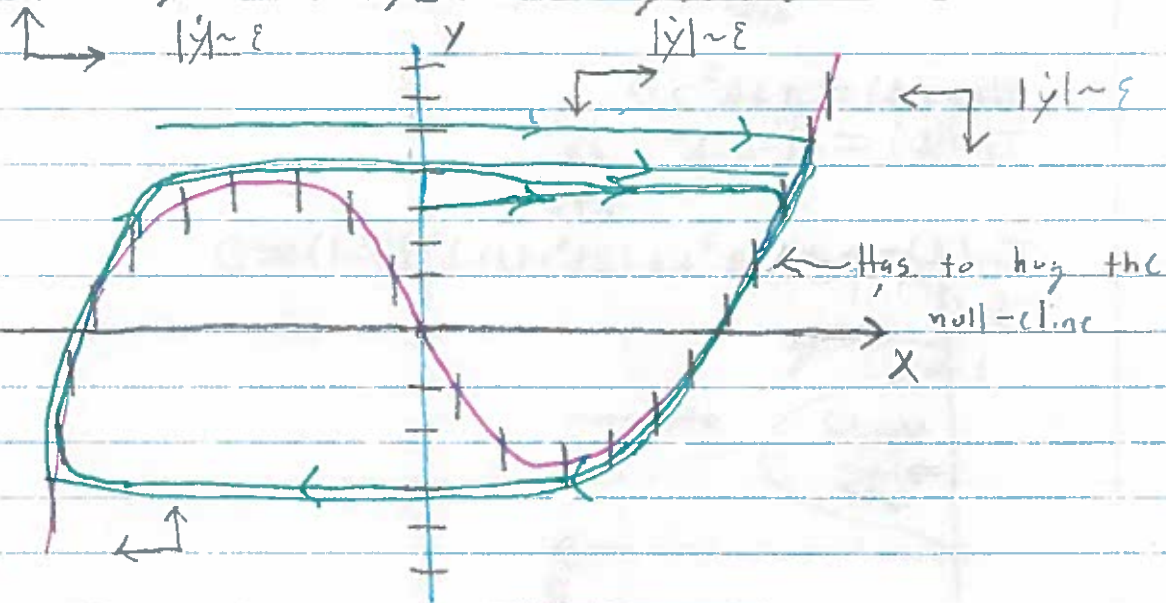
→ We expect the damping and pumping to lead to oscillations.

$$\ddot{x} + \frac{1}{\epsilon}(x^2 - 1)\dot{x} = \frac{d}{dt} \left(\dot{x} + \frac{1}{\epsilon} \left(\frac{x^3}{3} - x \right) \right)$$

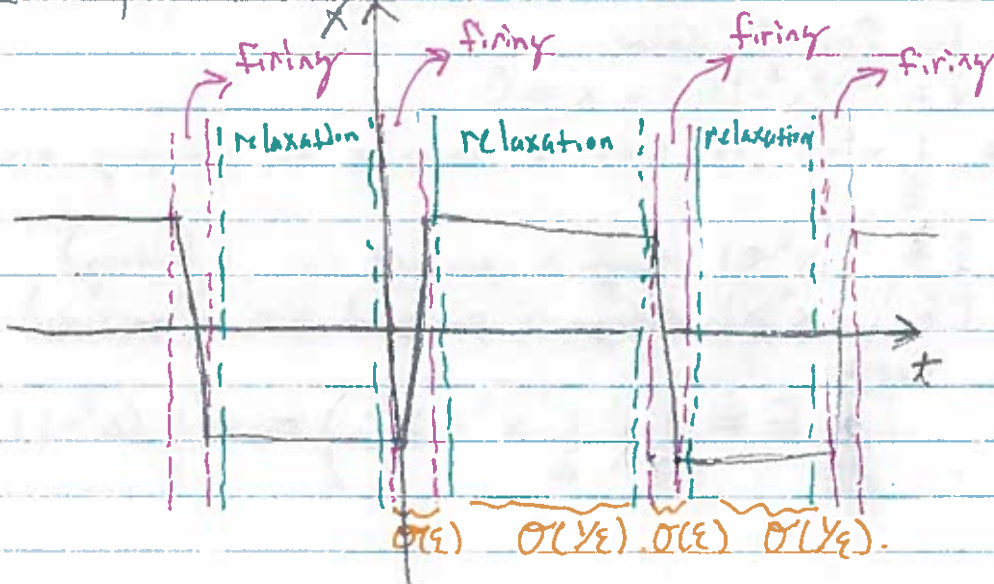
Let $F(x) = \frac{x^3}{3} - x$, $y = \epsilon \dot{x} + \frac{x^3}{3} - x$

$$\Rightarrow \begin{cases} \dot{x} = \frac{1}{\epsilon}(y - F(x)), & 0 < \epsilon \ll 1 \\ \dot{y} = -\epsilon x \end{cases}$$

Lets try to analyze this system:



Lets plot the dynamics of x'



Can we estimate the time scales. Essentially:

$$\dot{x} \sim 1/\varepsilon \text{ away from the null-cline.}$$

At the null-cline

$$\dot{x} \sim \varepsilon, \dot{y} \sim \varepsilon$$

as the two terms balance each other

Example:

$$\ddot{x} = (1 - 3x^2 - 2\dot{x}^2)\dot{x} - x$$

Let $v = \dot{x}$,

$$\dot{x} = v$$

$$\dot{v} = (1 - 3x^2 - 2v^2)v - x$$

The phase portrait analysis is done in Mathematica.
How can we prove there is a limit cycle?

Convert to polar coordinates!

$$\dot{r} = \cos\theta \dot{x} + \sin\theta \dot{v}$$

$$\dot{r} = \cos\theta \cdot r \sin\theta + [(1 - 3\cos^2\theta r^2 - 2r^2 \sin^2\theta) r \sin\theta - r \cos\theta] \sin\theta$$

$$\Rightarrow \dot{r} = \sin\theta [r \cos\theta + (1 - r^2 - \cos^2\theta r^2) r \sin\theta - r \cos\theta]$$

$$\dot{r} = r \sin^2\theta [1 - 2r^2 - r^2 \cos^2\theta]$$

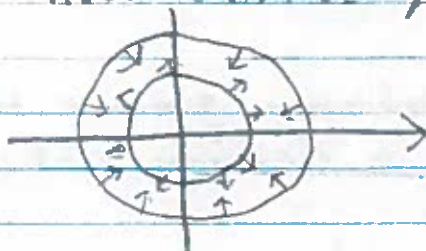
Let $r = 1/2$

$$\text{Then } \dot{r} = 1/2 \sin^2\theta [1 - 1/2 - 1/2 \cos^2\theta] \geq 0$$

Let $r = 1/\sqrt{2}$

$$\text{Then } \dot{r} = 1/\sqrt{2} \sin^2\theta [-1/2 \cos^2\theta] \leq 0$$

Therefore, we have constructed a trapping region.



\Rightarrow A limit cycle exists!

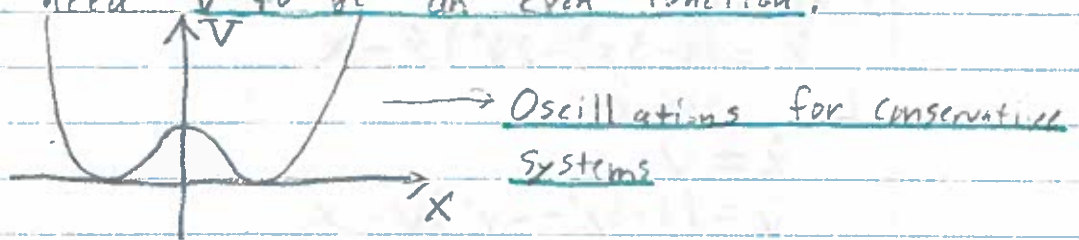
Lienard Systems

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

damping force from external
potential $V(x) = + \int_{x_0}^x g(x) dx$.

We saw from the Van-der pol oscillator that we need several conditions for a limit cycle.

1. We need V to be an even function.



2. We need V to satisfy $\lim_{|x| \rightarrow \infty} V(x) = \infty$.

→ Trajectories are "trapped" for conservative system.

Rewrite

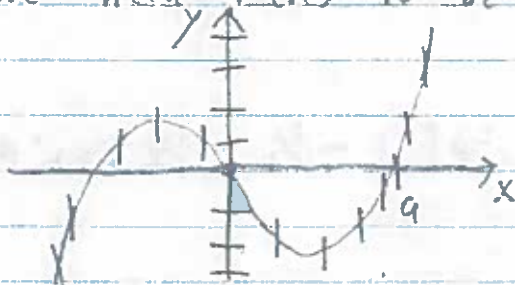
$$\frac{d}{dt}(x + F(x)) = -g(x), \quad F(x) = \int_{x_0}^x f(x) dx$$

Let

$$y = \dot{x} + F(x)$$

$$\Rightarrow \begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$$

3. We need $F(x)$ to be odd:



4. We need $F(x)$ to satisfy:

$F(x)$ is increasing on \mathbb{R}^+ and, $\exists a \in \mathbb{R}$ such that $F(x) > 0$ if $x > a$

* These four conditions guarantee the existence of a limit cycle. This is a sufficient condition.