

Chapter 9: Lorenz Equations.

3-D dynamics.

Old tools:

1. local linearization

2. global analysis

* Trajectories can be attracted to lower dimensional subsets:

Surfaces:

a.) Spheres

b.) Torus.

Example

$$\dot{x} = -y + \frac{x(1-x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{y} = x + \frac{y(1-x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{z} = \frac{z(1-x^2-y^2-z^2)}{\sqrt{x^2+y^2+z^2}}$$

Convert to spherical polar:

$$\dot{\rho} = 1 - \rho^2$$

$$\dot{\theta} = 1$$

$$\dot{\phi} = 0$$



* All trajectories go to sphere of radius 1 and form limit cycles of constant angle ϕ .

* Approach sphere along a cone.

Example

$$\dot{x} = \frac{xz}{\sqrt{x^2+y^2}} + \frac{x(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{y} = \frac{yz}{\sqrt{x^2+y^2}} + \frac{y(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

$$\dot{z} = \frac{z(1-(x^2+y^2+z^2))}{\sqrt{x^2+y^2+z^2}}$$

Convert to spherical Polar

$$\dot{\rho} = 1 - \rho^2$$

$$\dot{\theta} = 0$$

$$\dot{\phi} = 1$$

Converge to latitudes \rightarrow This violates existence and uniqueness
This occurs because x^2+y^2 can equal 0.

Example:

$$\dot{r} = (1-r^2)$$

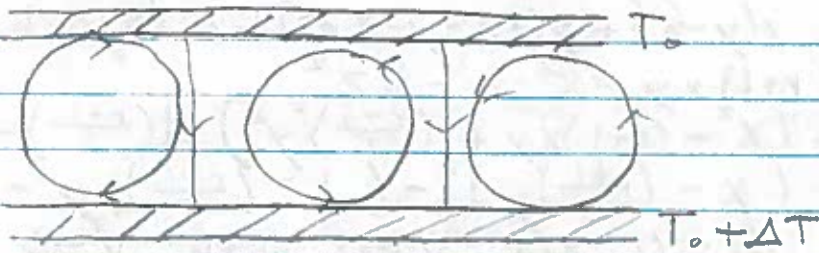
$$\dot{\theta} = 1$$

$$\dot{\phi} = 1$$

This forms Lissajous cycles.

* In these examples the long term behaviour is attracted to surfaces which we can analyze dynamics on

Lorenz Equations:



$$\dot{x} = \sigma(y - x) \rightarrow \text{Convective motion}$$

$$\dot{y} = r x - y - x z \rightarrow \text{Temperature difference ascending/descending}$$

$$\dot{z} = x y - b z \rightarrow \text{distance from laminar}$$

σ - Prandtl number \rightarrow ratio of $\frac{\text{viscosity}}{\text{thermal diffusivity}}$

r - Rayleigh number: ($r \ll 1$ conduction, $r \gg 1$ convection)

b - dimensionless aspect ratio.

Next stuff happens as we play with r :

Fixed Points:

1. $(0, 0, 0) \rightarrow$ Pure conduction

2. $(\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1) \rightarrow$ left and right moving rolls



$$J(0, 0, 0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\begin{aligned} 2 \lambda_{1,2} &= -\sigma - 1 \pm \sqrt{(\sigma+1)^2 - 4(r - \sigma r)} \\ &= -\sigma - 1 \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)} \\ &= -\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma r} \end{aligned}$$

Case 1:

$r < 1$, only one fixed point and it is a stable node.

How can we eliminate closed orbits? Construct a Lyapunov function.

Let $V = \frac{1}{2} \left(\frac{1}{r} x^2 + y^2 + z^2 \right)$

$\Rightarrow \dot{V} = \frac{1}{r} x \dot{x} + y \dot{y} + z \dot{z}$

$= x(y-x) + y(rx-y-xz) + z(xy-bz)$

$= (r+1)xy - x^2 - y^2 - bz^2$

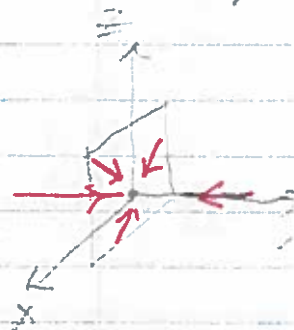
$= - \left(x^2 - (r+1)xy + \left(\frac{r+1}{2} \right)^2 y^2 \right) + \left[\left(\frac{r+1}{2} \right) - 1 \right] y^2 - bz^2$

$= - \left(x - \left(\frac{r+1}{2} \right) y \right)^2 - \left(1 - \left(\frac{r+1}{2} \right) \right) y^2 - bz^2$

If $r < 1$ $\dot{V} < 0$, and $V \geq 0$ with $V = 0$ if and only if $x = y = z = 0$.

$\Rightarrow \lim_{t \rightarrow \infty} V(t) = 0$.

If $r < 1$, all trajectories go to the origin



Case 2:

$r > 1$

$(0, 0, 0)$ has eigenvalues $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$

\rightarrow This is like a saddle point

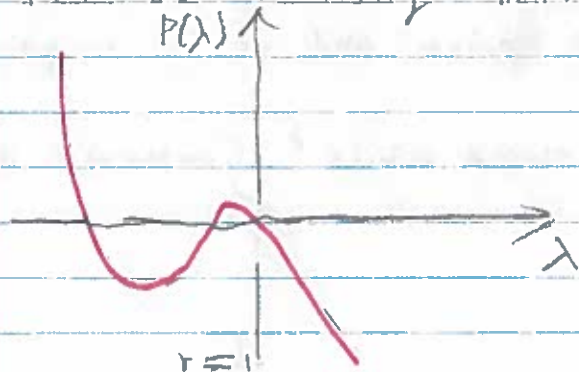
$(\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1)$ now exist \Rightarrow pitchfork bifurcation

We should do local analysis.

From now on fix $b = \frac{8}{3}, r = 10$. We denote new fixed points by C^+ and C^-

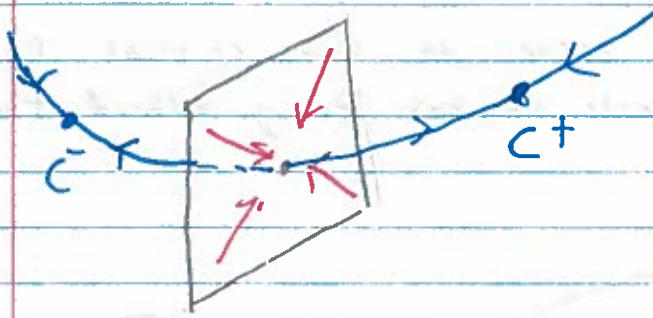
The characteristic polynomial is given by

$3P(\lambda) = -\lambda^3 - 41\lambda^2 - 88\lambda - 80(r-1)$. Plotting this can tell us interesting behavior



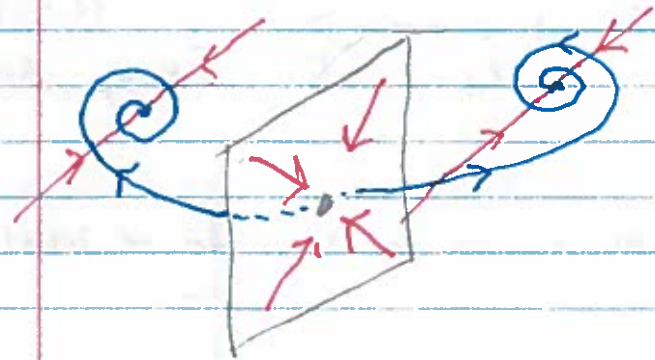
\Rightarrow Changing r shifts this graph down

\Rightarrow There exists r_s such that if $1 < r < r_s$ C_+, C_- are stable



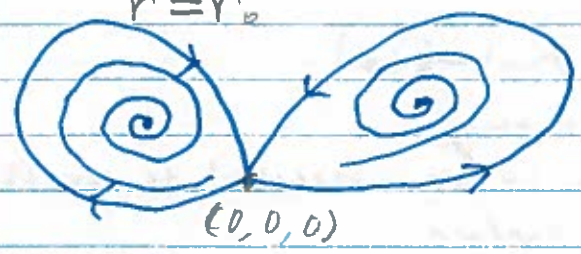
⇒ System has two steady state convection rolls.

Case 3:
 $r_3 < r < r_0$



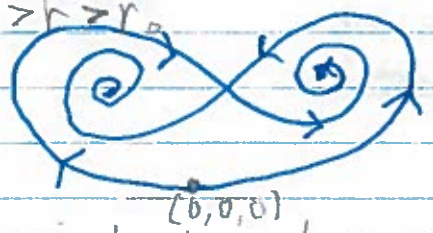
stability analysis tells us that they are stable spirals around the fixed points.

Case 4:
 $r = r_0$



→ Unstable limit cycle born in a homoclinic bifurcation.

Case 5:
 $r_4 > r > r_0$

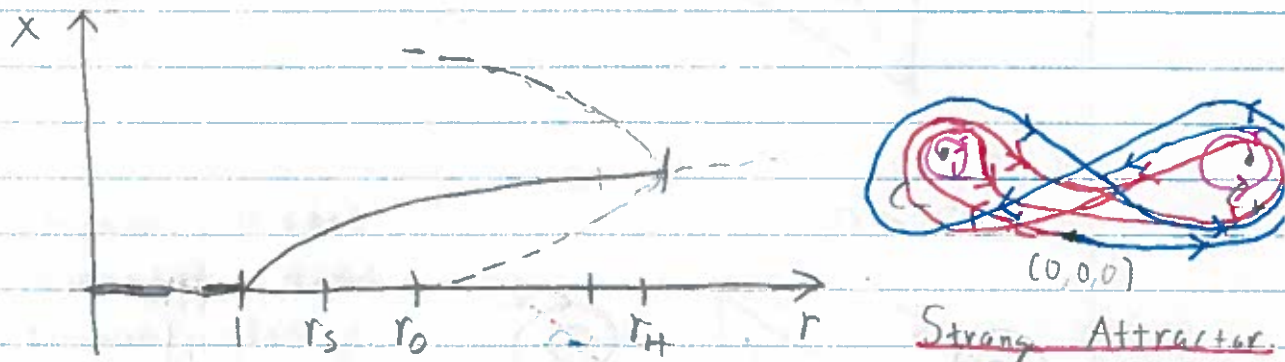


→ This creates a strange invariant set

spirals may loop around a lot. Impossible to predict which fixed point we go to

Case 6:

A Hopf bifurcation occurs at some critical r_H . The unstable limit cycle vanishes leaving behind two unstable spirals.



Volume Contraction:

$\vec{x} = F(\vec{x})$, let V_0 denote an initial volume of initial conditions

$$\Rightarrow \dot{V} = \int_{\partial S} \vec{F} \cdot \vec{n} dA$$

$$= \int_V \nabla \cdot \vec{F} dV$$

$$= \int_V (-\sigma - 1 - b) dV$$

$$= (-\sigma - 1 - b)V$$

$$\Rightarrow V(t) = V_0 \exp(-(\sigma - 1 - b)t)$$

All volumes shrink to nothing.

\Rightarrow The trajectories must be attracted to something...

\rightarrow 1. can't be a surface

2. can't be a fixed point (all unstable if $r > r_H$)

3. possible to eliminate limit cycles.

4. can't be quasiperiodic

\rightarrow Must be a new object!

\downarrow

Goals!

1. Find out what new object is.

2. Try to study chaos.

What is Chaos?

* Chaos - Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic - There exist trajectories which do not go to fixed points, periodic orbits, or quasi-periodic orbits.
2. Sensitive Dependence - We say $\vec{x} = F(\vec{x})$ has sensitive dependence on initial data on a set A if $\forall \vec{x}_0 \in A, \exists \epsilon > 0$ such that $\forall \delta > 0, \exists \vec{x}_0$ and $T > 0$ with $|\vec{x}_0 - \tilde{x}| < \delta$ and $|\vec{x}(T) - \tilde{x}(T)| > \epsilon$

Trajectories remain bounded

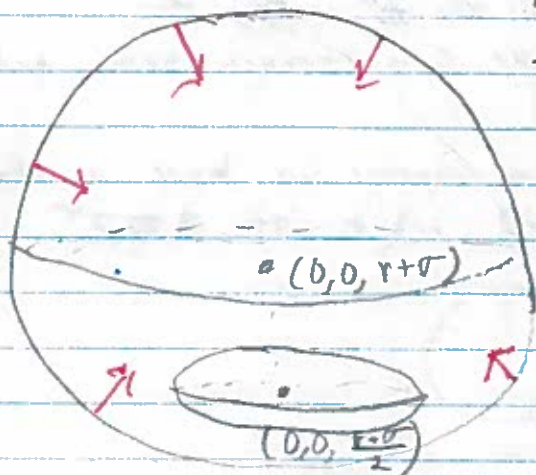
Consider a spherical surface $S_r = x^2 + y^2 + (z - r - \sigma)^2 = R^2$

For a trajectory starting on this surface we have that

$$\begin{aligned} \frac{d}{dt} [x^2 + y^2 + (z - r - \sigma)^2] &= 2x\dot{x} + 2y\dot{y} + 2(z - r - \sigma)\dot{z} \\ &= -2 \left[a x^2 + y^2 + b \left(z - \frac{r + \sigma}{2} \right)^2 - \frac{b(r + \sigma)^2}{4} \right] \end{aligned}$$

Pick R large enough that the ellipse $a x^2 + y^2 + b \left(z - \frac{r + \sigma}{2} \right)^2 = \frac{b(r + \sigma)^2}{4}$

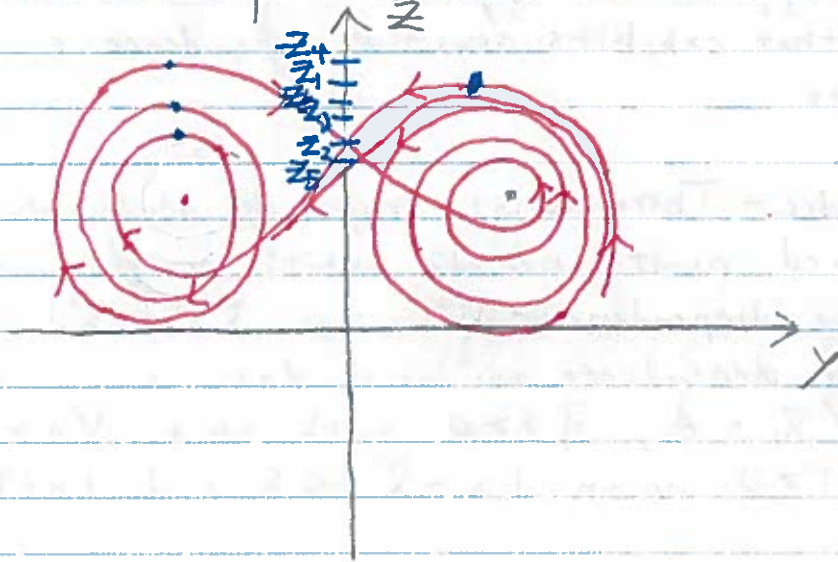
is enclosed in S_r . This guarantees S_r is a trapping region



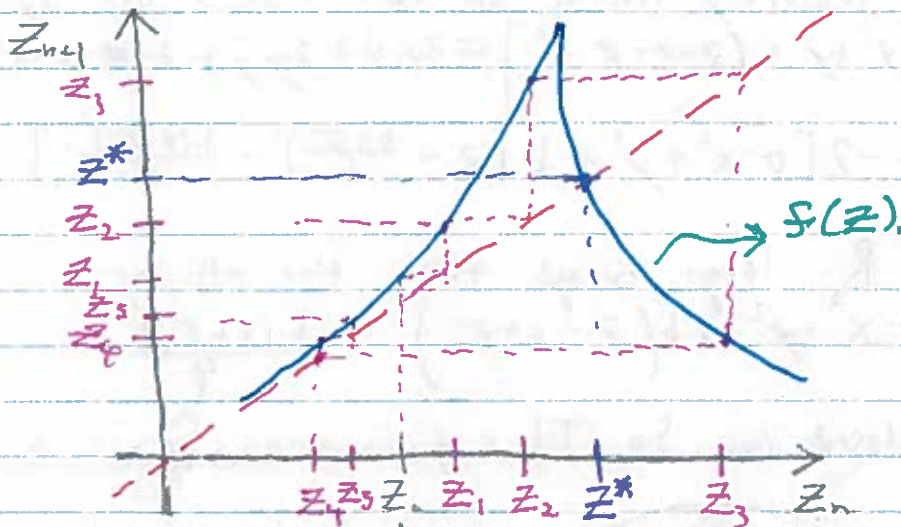
\Rightarrow Trajectories remain bounded and must go to something.

Lorenz Map

Take a particular view of a trajectory

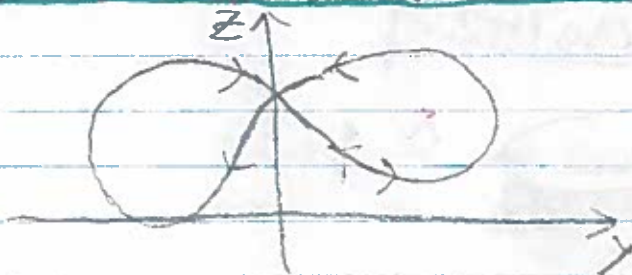


The Lorenz map looks at the relationship between z_n and z_{n+1}



This is awesome! We can extract some order from chaos
 $z_{n+1} = f(z_n)$.

Note $|f'(z)| > 1$. Now suppose we have a stable limit cycle.
There is one closed orbit at $z = z^*$



Consider

$$z_0 = z^* + \delta_0$$

$\delta_0 =$ initial perturbation.

$$\Rightarrow z_1 = f(z^* + \delta_0)$$

$$\Rightarrow f(z) \approx f(z^*) + f'(z^*)(z - z^*)$$

$$\Rightarrow z_1 = f(z_0) \approx z^* + f'(z^*)\delta_0$$

$$\Rightarrow \delta_1 = z_1 - z^* \approx f'(z^*)\delta_0, \quad \delta_1 = z_1 - z^*$$

$$\Rightarrow |\delta_1| > |\delta_0|$$

The trajectory grows away from the set. This fixed point is unstable.

\Rightarrow No stable limit cycles.

Attractor

An attractor is a closed set A with the following properties.

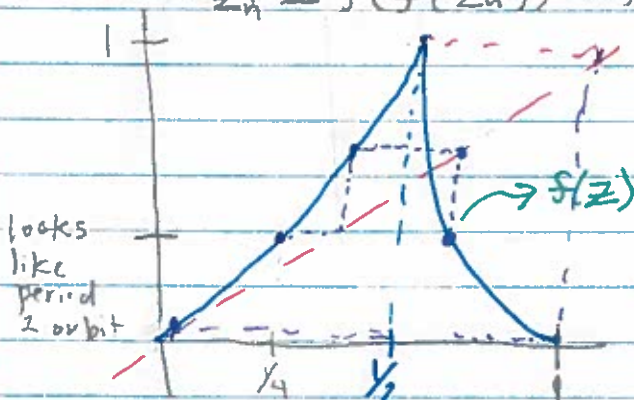
1. A is invariant: Any $\vec{x}(t)$ that starts in A remains in A .

2. A attracts an open set of initial conditions: \exists exists an open set U containing A such that if $\vec{x}(0) \in U$, then $d(\vec{x}(t), A) \rightarrow 0$ as $t \rightarrow \infty$. The largest U is called the basin of attraction.

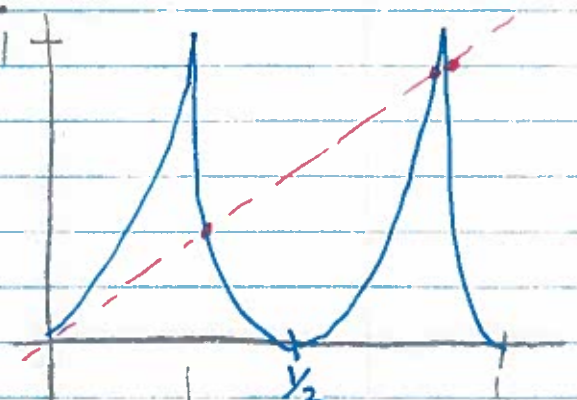
3. A is minimal: There is no proper subset of A

What about Double Periodic Iterations?

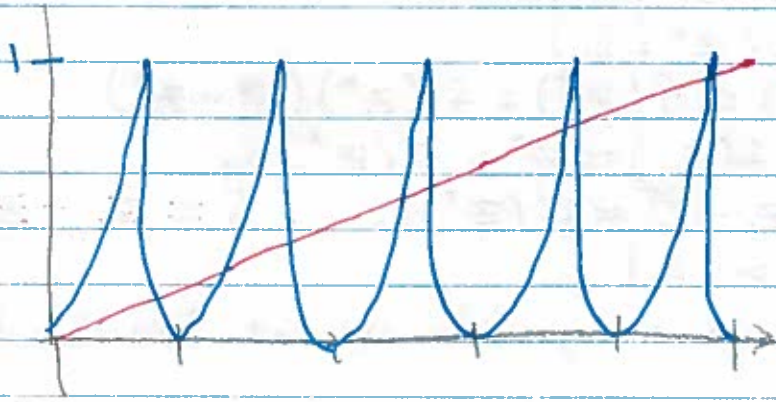
$$z_n = f(f(z_n)) = f^2(z_n)?$$



$$f^2(1/2) \approx 0, \quad f^2(1/4) \approx 1/4.$$



General Case



We can continue forever

→ There exists an infinite number of unstable limit cycles

