

Math 383/683
Spring 2017
Exam 1
2/10/17

Name (Print): Key.

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a “fundamental theorem” you must indicate this and explain why the theorem may be applied.**
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
6	20	
7	20	
Total:	100	

1. (10 points) (**True or False**) Identify whether the following statements are true or false. If a statement is false provide a counterexample or explain why you think the statement is false.

(a) (2 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical system $\dot{x} = f(x)$. If $x(t)$ is a solution to this dynamical system then $\ddot{x} = f'(x)$.

False, $\ddot{x} = f'(x)f(x)$.

(b) (2 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be smooth and consider the dynamical system $\dot{x} = f(x)$. If this system has two fixed points then these two fixed points can both be stable.

False, one root must be stable or semistable.

(c) (2 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the dynamical $\dot{x} = f(x)$ which has only one fixed point x^* and it is stable. If $x(t)$ solves $\dot{x} = f(x)$ with $x(0) = x_0 \neq x^*$ then there exists T satisfying $0 < T < \infty$ such that $x(T) = x^*$.

False, $\lim_{t \rightarrow \infty} x(t) = x^*$, but $x(t) = x^*$ violates uniqueness.

(d) (2 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function. x^* is an unstable fixed point for the equation $\dot{x} = f(x)$ if and only if $f'(x^*) > 0$.

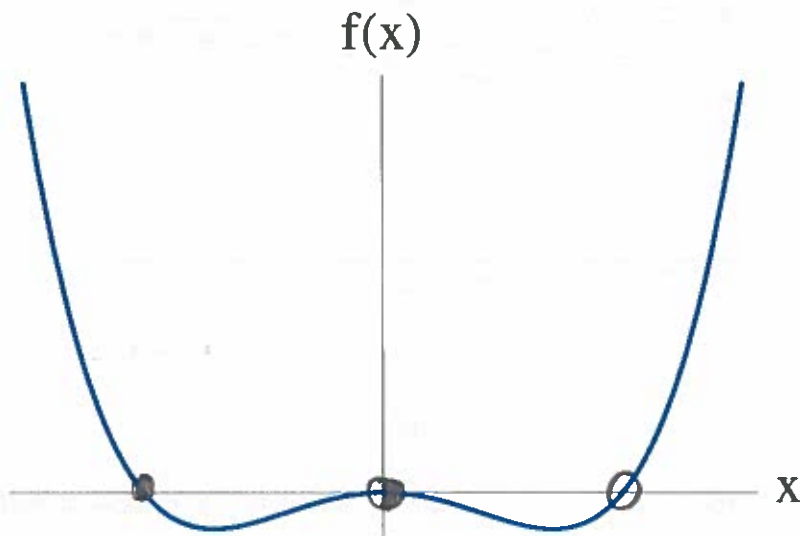
False, consider $\dot{x} = x^3$.

(e) (2 points) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function and consider the differential equation $\dot{x} = f(x)$ which has one fixed x^* which is unstable. If $x(t)$ solves this differential equation then $\lim_{t \rightarrow -\infty} x(t) = x^*$.

True.

2. (10 points)

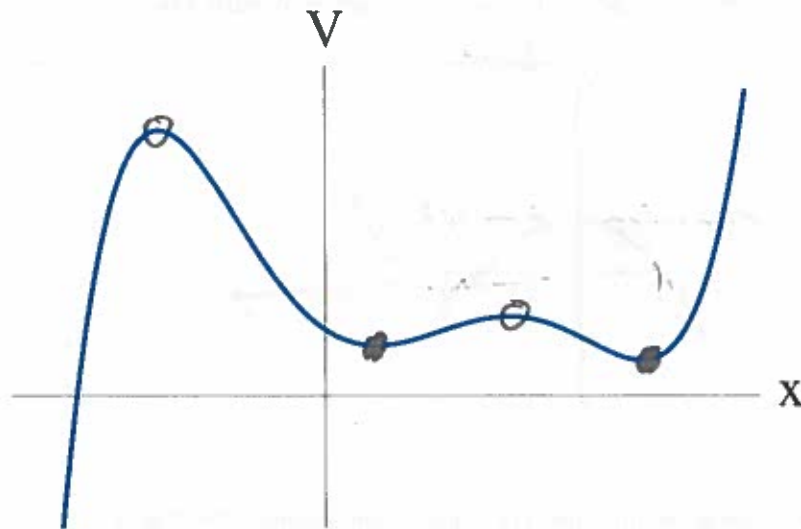
- (a) (5 points) Consider the differential equation $\dot{x} = f(x)$ with $x \in \mathbb{R}$ for the function $f(x)$ drawn below. On the graph below, indicate the equilibrium of this differential equation and their stability.



- (b) (5 points) Consider the differential equation

$$\dot{x} = -\frac{dV}{dx},$$

with $x \in \mathbb{R}$, where the potential $V(x)$ is drawn below. On the graph below, indicate the equilibrium of this differential equation and their stability.



3. (15 points) The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$.

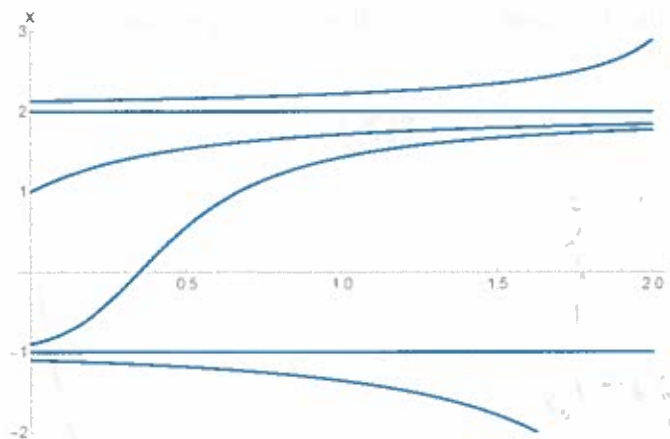
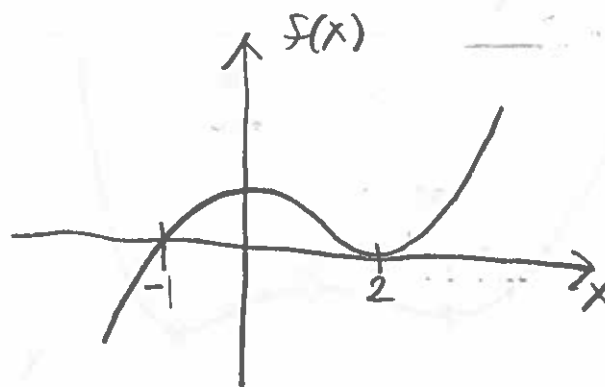


Figure 1:

- (a) (5 points) Sketch a one dimensional phase portrait that is consistent with Fig. 1.



- (b) (5 points) Sketch a graph of $f(x)$ that is consistent with Fig. 1.



- (c) (5 points) Give a formula for $f(x)$ that is consistent with Fig. 1.

$$f(x) = (x+1)(x-2)^2.$$

4. (10 points) The following question concerns the dynamics of a population of insects that are being preyed on by birds. The dynamics of the population of the insects $P \geq 0$ can be modeled by the equation:

$$\dot{P} = rP \left(1 - \frac{P}{\kappa}\right) - \frac{AP^2}{B^2 + P^2},$$

where $r, \kappa, A, B > 0$ are constants.

- (a) (5 points) What are the dimensions of the constants r, κ, A, B ?

$$[r] = \frac{1}{T}$$

$$[\kappa] = P$$

$$[B] = P$$

$$[A] = \frac{P}{T}$$

- (b) (5 points) Show by choosing appropriate scales that this equation can be put into the following dimensionless form:

$$\frac{dx}{d\tau} = \alpha x \left(1 - \frac{x}{\beta}\right) - \frac{x^2}{1+x^2},$$

where $\alpha, \beta > 0$ are constants.

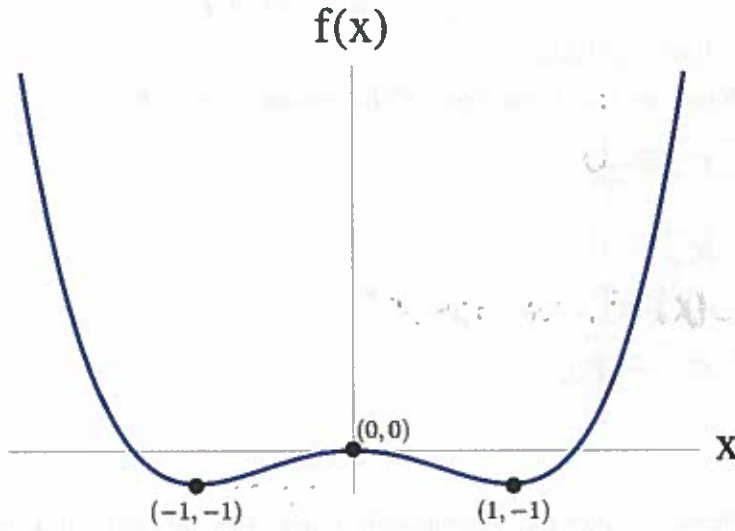
$$\text{Let } x = \frac{P}{B}, \tau = \frac{A}{B} t.$$

$$\frac{dP}{dt} = B \frac{dx}{dt} = B \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} = A \frac{dx}{d\tau}$$

$$A \frac{dx}{d\tau} = \alpha x \left(1 - \frac{x}{\beta}\right) - \frac{A B^2 x^2}{B^2(1+x^2)}$$

$$\Rightarrow \frac{dx}{d\tau} = \alpha x \left(1 - \frac{x}{\beta}\right) - \frac{x^2}{1+x^2}.$$

5. (15 points) Consider the following differential equation $\dot{x} = \mu + f(x)$ where $\mu \in \mathbb{R}$ is a constant and $f(x)$ is plotted below. Find and classify all bifurcation points of this differential equation and sketch the resulting bifurcation diagram. You can assume that the points are located at local maximum and minimum of $f(x)$.



Cases

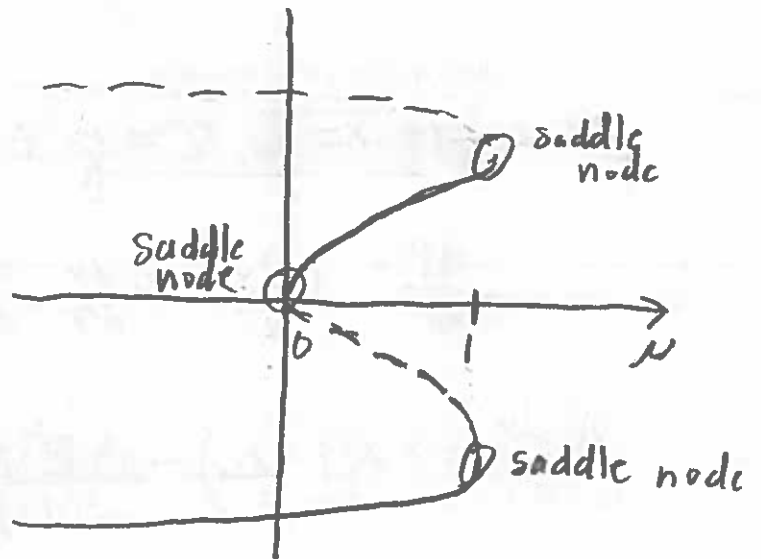
1. $\mu > 1$



2. $0 < \mu < 1$



3. $\mu < 0$



6. (20 points) For $\mu \in \mathbb{R}$, find and classify all bifurcation points of the differential equation

$$\dot{x} = x(\mu - 3) - x^2$$

and sketch the resulting bifurcation diagram.

10 Fixed Points

$$x(\mu - 3 - x) = 0$$

$$\Rightarrow x = 0, x = \mu - 3.$$

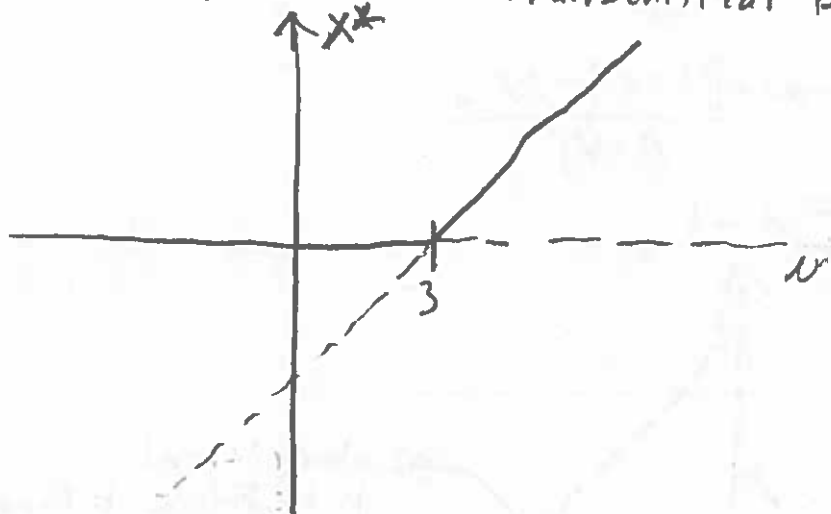
10 Stability:

$$\text{Let } f(x) = x(\mu - 3) - x^2$$

$$\Rightarrow f'(x) = \mu - 3 - 2x$$

$$\Rightarrow f'(0) = \mu - 3.$$

Therefore, there is a transcritical bifurcation at $\mu = 3$.



7. (20 points) For $\mu \in \mathbb{R}$, find and classify all bifurcation points of the differential equation

$$\dot{x} = \mu x - \frac{x}{1+x^2}.$$

and sketch the resulting bifurcation diagram.

5. Fixed Points:

$$x=0, \mu - \frac{1}{1+x^2} = 0 \Rightarrow 1+x^2 = \frac{1}{\mu}$$

$$\Rightarrow x = \pm \sqrt{\frac{1}{\mu} - 1}.$$

Pitchfork bifurcation at $\mu=1$.

5. Stability:

$$\text{Let } f(x) = \mu x - \frac{x}{1+x^2}.$$

$$\Rightarrow f'(x) = \mu - \frac{[(1+x^2) - 2x^2]}{(1+x^2)^2}$$

$$\Rightarrow f'(0) = \mu - 1$$

10.

