**Math 383/683**
**Spring 2017**
**Exam 2**
**3/31/17**

**Name (Print):** Key.

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This exam contains 9 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

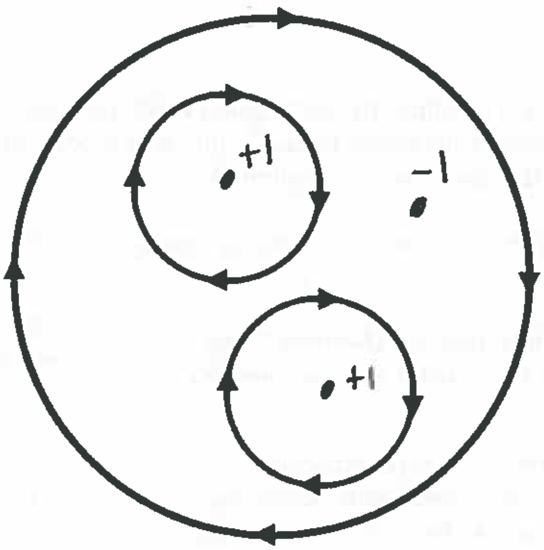
You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 20 | |
| 7 | 15 | |
| 8 | 10 | |
| 9 | 15 | |
| Total: | 100 | |

1. (5 points) A smooth vector field is known to have exactly three limit cycles as indicated below.




(a) (3 points) What is the minimum number of fixed points this system must contain?

(b) (2 points) For the minimum number of fixed points, how many of these fixed points can be spirals, nodes, or saddles?

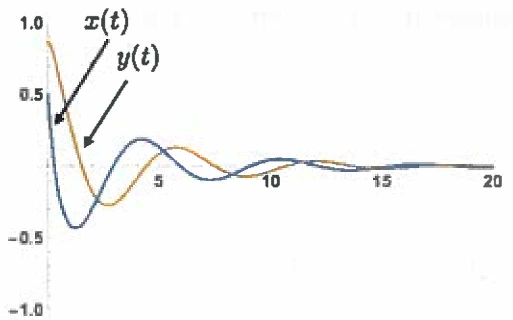
a.) 3

b.) 1 saddle, 2 unstable nodes or spirals.

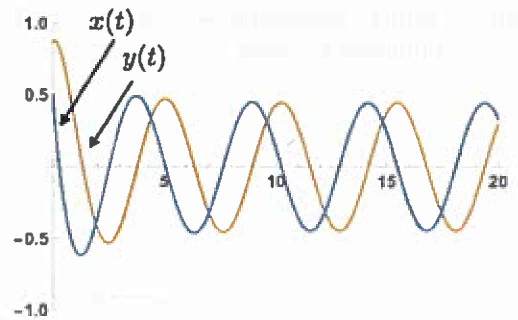
2. (5 points) Shown below are the graphs of two representative solutions of the system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases}$$

for $\mu < 0$ and $\mu > 0$. Identify the bifurcation that takes place at $\mu = 0$.




$\mu < 0$



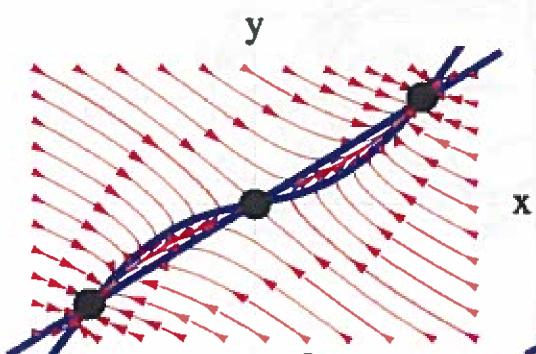

$\mu > 0$

Super-critical Hopf bifurcation.

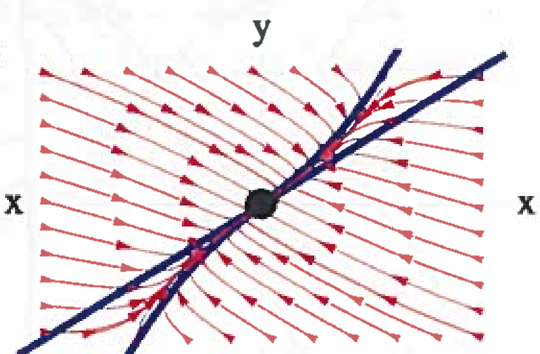
3. (5 points) Shown below are two phase portraits for the following system

$$\begin{cases} \dot{x} = f(x, y, \mu) \\ \dot{y} = g(x, y, \mu) \end{cases}$$

for $\mu < 0$ and $\mu > 0$. Identify the bifurcation that takes place at $\mu = 0$.




$\mu < 0$




$\mu > 0$

Super-critical pitchfork.

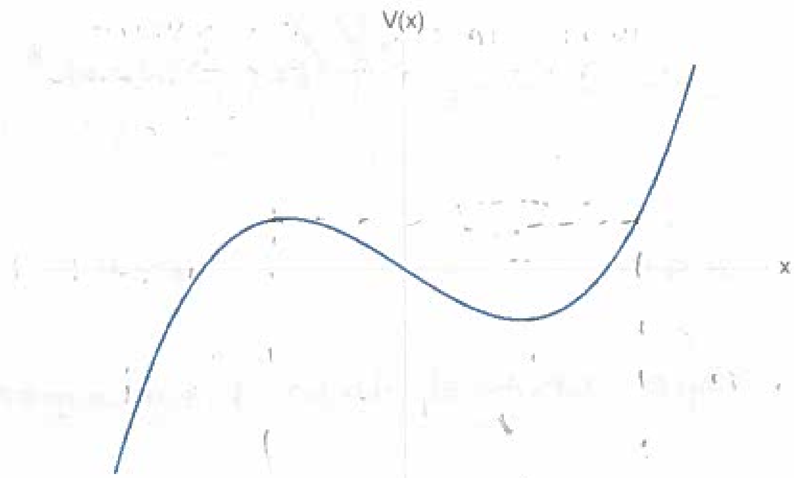
4. (15 points) Consider the system

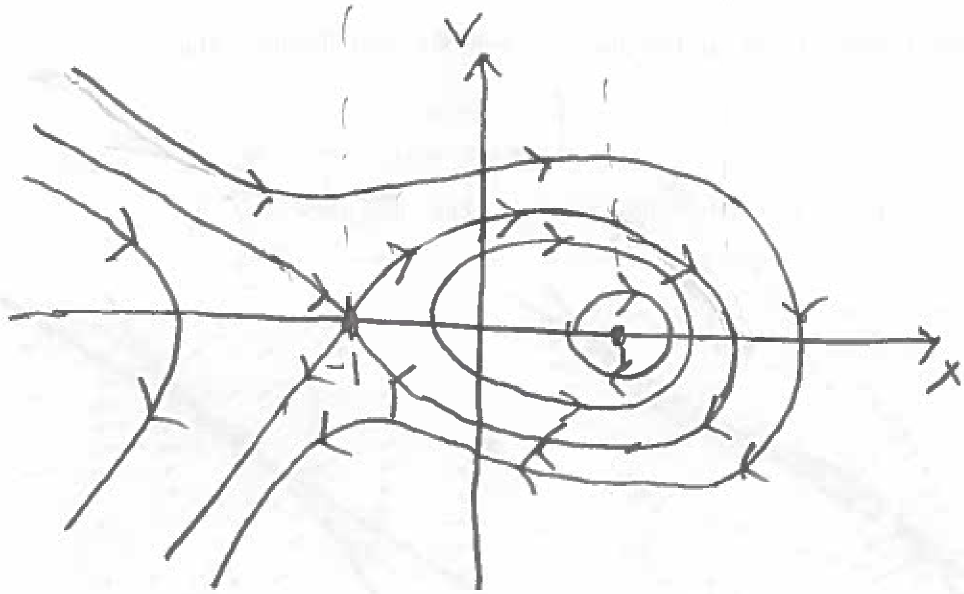
$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{dV}{dx} \end{cases},$$

where the graph of $V(x) = -x + x^3/3$ is plotted below.

(a) (10 points) Sketch a phase portrait for this system.

(b) (5 points) Determine an equation of the curves in $(x, v)$ coordinates for any heteroclinic or homoclinic orbits.




Along the homoclinic orbit we have from conservation of energy:

$$\frac{1}{2}v^2 + V(x) = \frac{1}{2}v_0^2 + V(x_0)$$

$$\Rightarrow \frac{1}{2}v^2 + \frac{x^3}{3} - x = +1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow \frac{1}{2}v^2 + \frac{x^3}{3} - x = \frac{2}{3},$$

where $x = -1$, $v = 0$

5. (10 points) Consider the dynamical system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases},$$

where $f, g : \mathbb{R}^2 \mapsto \mathbb{R}^2$ are smooth. If this system has a trapping region containing a single fixed point, prove that this fixed point cannot be a saddle.

Let R be the trapping region. By Poincare-Bendixon Solution curves must satisfy the following:

1. Solution curves approach stable limit cycle.
2. Solution curves approach stable fixed.

Since the fixed point is unstable only option 1 is possible. However, the index of a limit cycle is 1 and hence it cannot contain the saddle. Consequently, the fixed point cannot be a saddle.

6. (20 points) Consider the system

$$\begin{cases} \dot{x} = -x^2 + y \\ \dot{y} = x - y^2 \end{cases}.$$

(a) (5 points) Find and classify all equilibrium.

(b) (5 points) Plot the nullclines of the system.

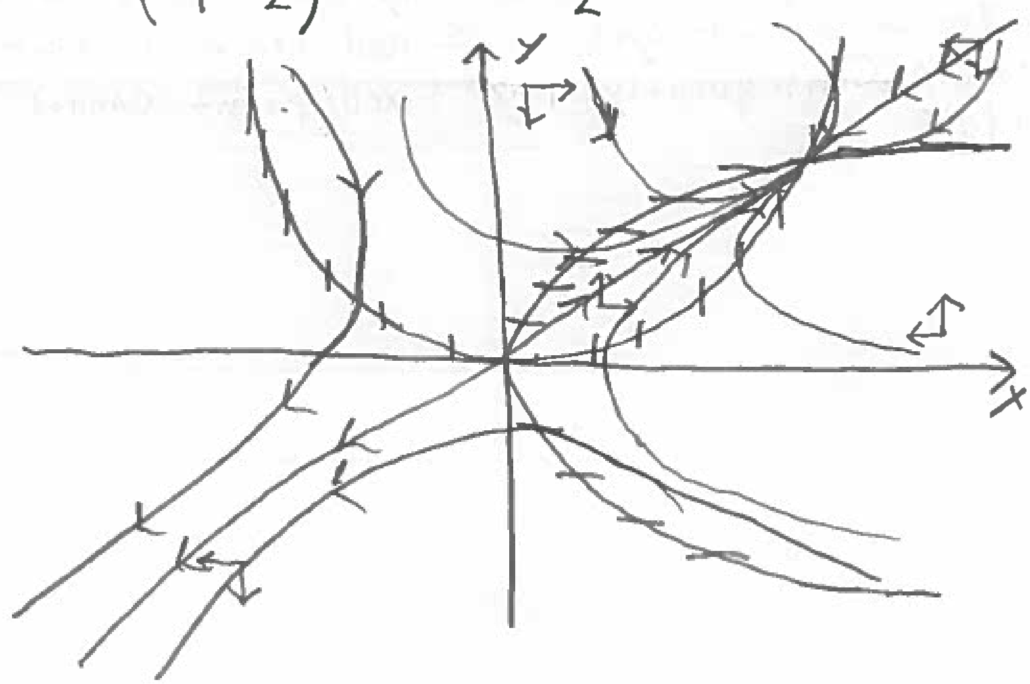
(c) (5 points) Sketch the phase portrait.

(d) (5 points) Show that the line $y = x$ is an invariant submanifold for this system. Can the system have any periodic orbits?

The equilibrium are $(1,1)$ and $(0,0)$.

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \text{saddle}$$

$$J(1,1) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-8}}{2} \Rightarrow \text{stable node.}$$




Let $s = y - x$. Then

$$\begin{aligned} \dot{s} = \dot{y} - \dot{x} &= x - y^2 + x^2 - y \\ &= x - y + (x-y)(x+y) \\ &= (x-y)(1 + (x+y)) \\ &= s(1 + (x+y)). \end{aligned}$$

Therefore, $s = 0$ is invariant $\Rightarrow y = x$ is invariant.

7. (15 points) Consider the following dynamical system on $\mathbb{R}^2$:

$$\begin{cases} \dot{x} = -x + 2y^3 - 2y^4 \\ \dot{y} = -x - y + xy \end{cases}.$$

(a) (10 points) Show that the function $L(x, y) = x^2 + y^4$ is a Lyapunov function for this system.

$$\dot{L} = 2x\dot{x} + 4y^3\dot{y}$$
$$= 2x(-x + \cancel{2y^3} - \cancel{2y^4}) + 4y^3(\cancel{-x} - y + \cancel{xy})$$
$$= -2x^2 - 4y^4$$
$$< 0.$$

Since $L \geq 0$, and $\dot{L} < 0$ this is a Lyapunov function.

(b) (5 points) Can this system have a limit cycle?

No.

8. (10 points) For each of the below dynamical systems defined on the torus, match them with the corresponding phase portrait on the square representation of the torus.

$\dot{\psi} = \dot{\theta} - \dot{\phi}$
$= -2\sin(\psi)$

$$(i) \begin{cases} \dot{\theta} = 1 + \sin(\phi - \theta) \\ \dot{\phi} = 1 + \sin(\theta - \phi) \end{cases}$$

$$(ii) \begin{cases} \dot{\theta} = 4 + \sin(\phi - \theta) \\ \dot{\phi} = 1 + \sin(\theta - \phi) \end{cases}$$

$\dot{\psi} = 3 - 2\sin(\psi)$

$\dot{\psi} = \sin(\psi)$

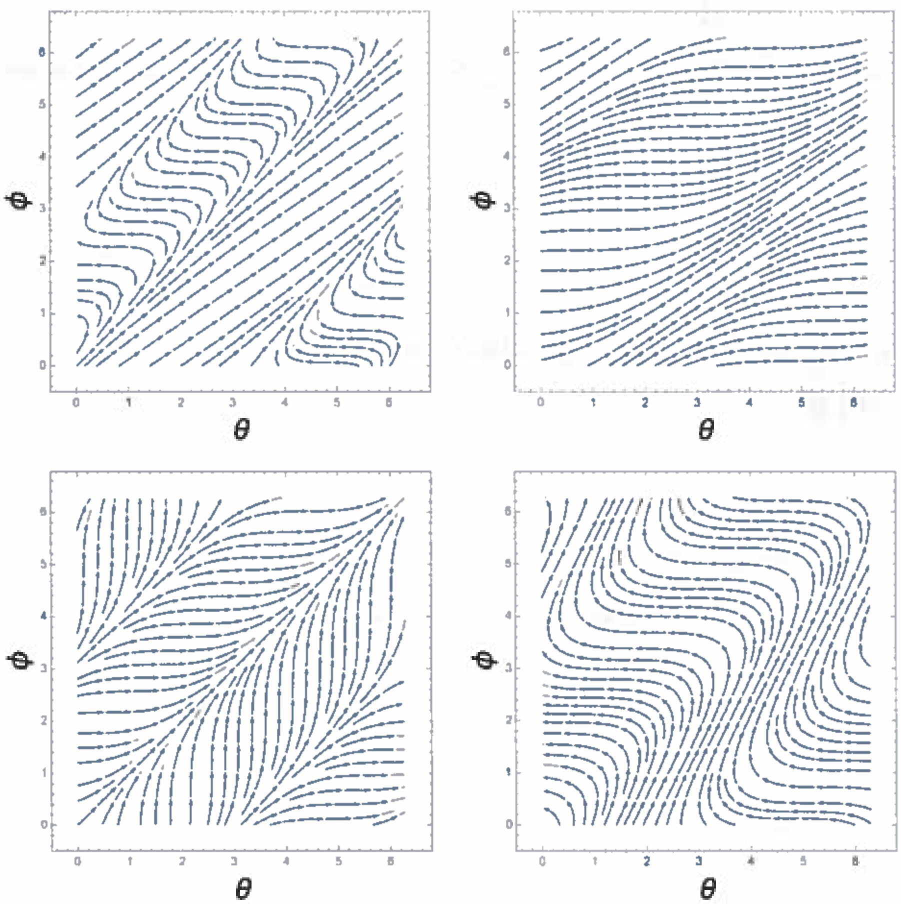
$$(iii) \begin{cases} \dot{\theta} = 1 + 2\sin(\theta - \phi) \\ \dot{\phi} = 1 + \sin(\theta - \phi) \end{cases}$$

$$(iv) \begin{cases} \dot{\theta} = -1 + 2\sin(\theta - \phi) \\ \dot{\phi} = 1 + \sin(\theta - \phi) \end{cases}$$

$\dot{\psi} = -2 + \sin(\psi)$

(i)

(ii)




(iii)

(iv)

9. (15 points) Consider the system

$$\begin{cases} \dot{x} = y + \mu x + x^3 \\ \dot{y} = \mu y - x + x^2 y \end{cases},$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) (10 points) Show that a Hopf bifurcation occurs in this system as $\mu$ is varied.

(b) (5 points) Determine if the Hopf bifurcation is subcritical or supercritical. **Converting to polar coordinates might be useful.**

$$J(0,0) = \begin{pmatrix} \nu & 1 \\ -1 & \nu \end{pmatrix} \Rightarrow \lambda = \frac{-2\nu \pm \sqrt{4\nu^2 - (\nu^2+1)}}{2}$$

Clearly a Hopf bifurcation occurs when $\nu = 0$.

$$r\dot{r} = x\dot{x} + y\dot{y}$$

$$\Rightarrow r\dot{r} = \cancel{xy} + \nu x^2 + x^4 + \nu y^2 - \cancel{xy} + x^2y^2$$

$$= \nu r^2\cos^2\theta + r^4\cos^4\theta + \nu r^2\sin^2\theta + r^4\cos^2\theta\sin^2\theta$$

$$= \nu r^2 + r^4\cos^2\theta$$

Since $\dot{r} > 0$ if $\nu > 0$ this must be a sub-critical Hopf bifurcation.