Homework #1

#2.1.3

Consider the system $\dot{x} = \sin(x)$.

a.) Find $\ddot{x}$ as a function of x.

b.) Find the points x where $\ddot{x}$ is maximized.

Solution

a.) If $x(t)$ satifies $\dot{x} = \sin(x)$ then

$$\ddot{x} = \frac{d}{dt}\sin(x(t))$$

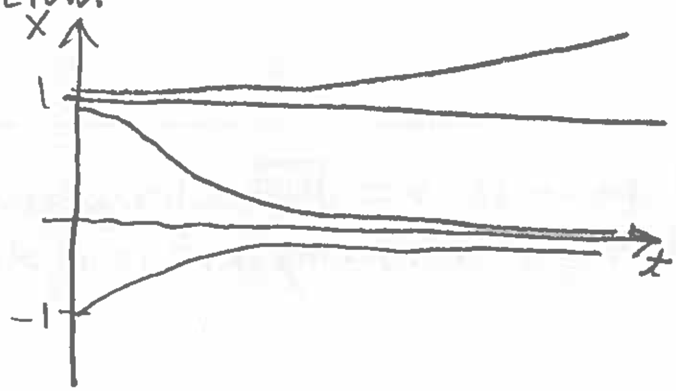
$$= \cos(x(t)) \cdot \dot{x}$$

$$= \cos(x(t)) \cdot \sin(x).$$

b.) Since $\ddot{x} = \cos(x) \cdot \sin(x) \leq 1/2$ it follows that $\ddot{x}$ is maximized when $\cos(x) \cdot \sin(x) = 1/2$, e.g. $x = \pi/4 + n\pi$ where $n \in \mathbb{Z}$.

#2.2.9

Find an equation $\dot{x} = f(x)$ whose solutions are consistent with those shown below.



Solution:

$$\dot{x} = x(x-1).$$

#2.2.13

The velocity of a skydiver is modeled by

$$m\dot{v} = mg - kv^2,$$

where $m, g, k > 0$.

a.) Obtain an analytical solution assuming $v(0) = 0$.

b.) Calculate $\lim_{t\to\infty} v(t)$.

c.) Give a graphical analysis of the problem.

Solution:

a.) Seperating variables it follows that:

$$\int_0^v \frac{1}{g - \frac{k}{m}v^2}\,dv = \int_0^t dt$$

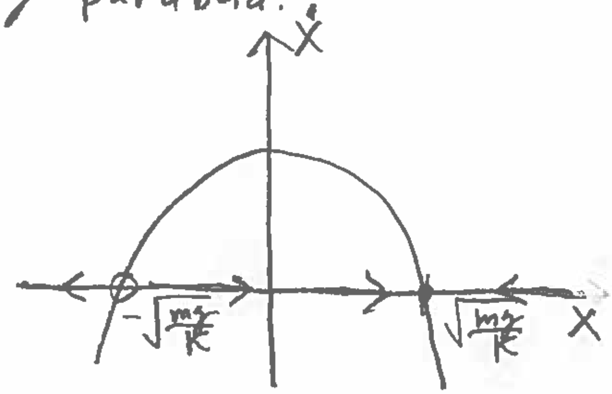
$$\Rightarrow \frac{1}{g}\int_0^v \frac{1}{1 - \left(\sqrt{\frac{k}{mg}}v\right)^2}\,dv = \int_0^t dt.$$

$$\Rightarrow \frac{1}{g}\sqrt{\frac{mg}{k}}\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right) = t.$$

$$\Rightarrow v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right).$$

b.) $\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}}$.

c.) Geometrically, the only fixed point is $v = \sqrt{\frac{mg}{k}}$. Clearly this point is stable since the function $f(v) = mg - kv^2$ is a downward opening parabola!



#2.3.5

Suppose X, Y are two species that reproduce exponentially fast:

$$\dot{X} = aX,$$
$$\dot{Y} = bY$$

where $a > b > 0$.

a.) Let $x(t) = X(X+Y)^{-1}$. By solving for X and Y show that $\lim_{t\to\infty} x(t) = 1$.

b.) Show that x satisfies the logistic equation. Explain why this implies $\lim_{t\to\infty} x(t) = 1$.

Solution:

a.) If $\dot{X} = aX$ and $\dot{Y} = bY$ then $x(t) = x_0 \exp(at)$, $y(t) = y_0 \exp(bt)$. Therefore,

$$x(t) = \frac{x_0 e^{at} + y_0 e^{bt}}{x_0 e^{at}}$$

$$\Longrightarrow \lim_{t\to\infty} x(t) = \lim_{t\to\infty} \left(1 + \frac{y_0}{x_0} e^{-(a-b)t}\right) = 1.$$

b.) Differentiating it follows that:

$$\begin{aligned}
\dot{x} &= \dot{X}(X+Y)^{-1} - X(X+Y)^{-2}(\dot{X}+\dot{Y}) \\
&= aX(X+Y)^{-1} - X(X+Y)^{-2}(aX+bY) \\
&= \frac{aX^2 + aXY - aX^2 - bXY}{(X+Y)^2} \\
&= \frac{(a-b)X \cdot Y}{(X+Y)^2} \\
&= (a-b)x(1-x).
\end{aligned}$$

Consequently,

$$\lim_{t\to\infty} x(t) = 1.$$

$\blacksquare$

#2.3.6.

Consider the model

$$\dot{x} = s(1-x)x^a - (1-s)x(1-x)^a,$$

where $0<s<1$, $a>1$, $0 \le x \le 1$.

a.) Show that this equation has three fixed points.

b.) Show that for all $a>1$, the fixed points $x=0,1$ are stable.

c.) Show that the third fixed point, $0<x^*<1$, is unstable.

Solution:

a.) $\dot{x} = (1-x)\cdot x(s x^{a-1} - (1-s)(1-x)^{a-1})$.

Clearly, $x=0,1$ are fixed points. Let $g(x) = s x^{a-1} - (1-s)(1-x)^{a-1}$; Since $g(0) = -(1-s) < 0$ and $g(1) = s > 0$ it follows that there exists a third root $x^* \in (0,1)$. Moreover, for $x \in [0,1]$ it follows:

$$g'(x) = s(a-1)x^{a-2} + (1-s)(1-x)^{a-2} > 0.$$

Consequently, x^* is unique.

b.) Let $f(x) = s(1-x)x^a - (1-s)x(1-x)^a$ and $y = (1-x)$. Calculating it follows that:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[s(1-x)x^a - (1-s)x(1-x)^a\right] \\ &= \frac{d}{dx}\left[s(x^a - x^{a+1})\right] + \frac{d}{dy}\left[(1-s)(1-y)y^a\right] \\ &= s\left[a x^{a+1} - (a+1)x^a\right] + (1-s)\left[a y^{a-1} - (a+1)y^a\right] \end{aligned}$$

Therefore,

$$\begin{aligned} f'(0) &= (1-s)[a-(a+1)] \\ &= -(1-s) < 0 \\ f'(1) &= -s < 0. \end{aligned}$$

Consequently, $x=0,1$ are stable fixed points.

c.) Since $f(x)$ is smooth and x^* is the unique fixed point in the interval $(0,1)$ it follows that x^* must be unstable since $0,1$ are stable fixed points.