Homework #2.

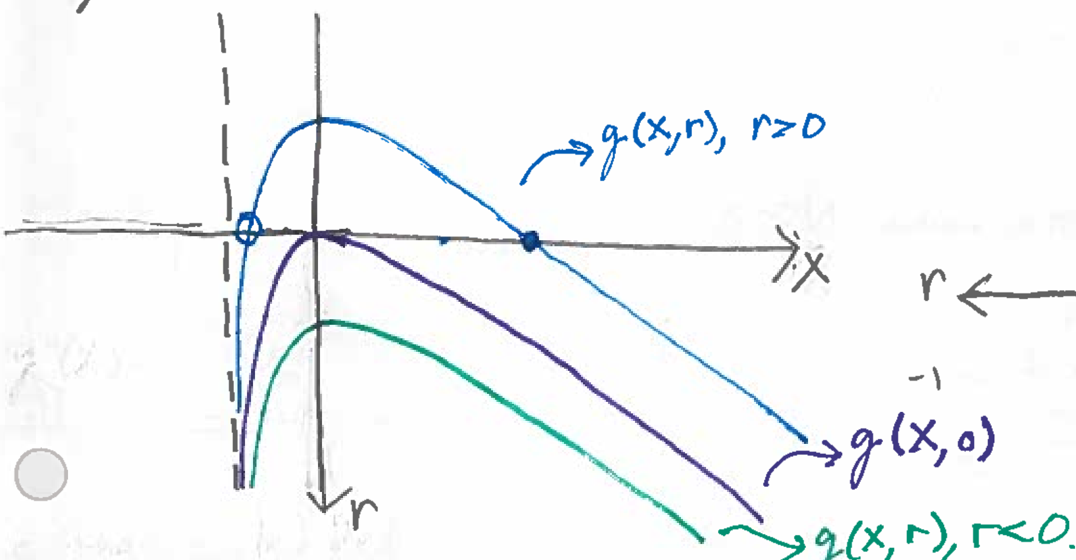
#3.1.3.

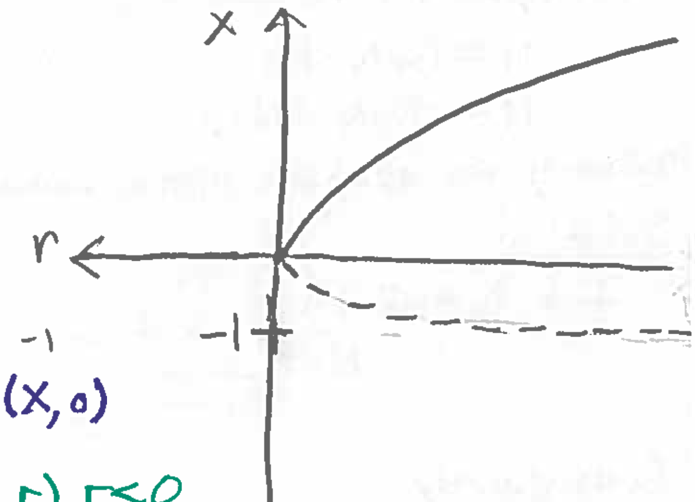
Show that a saddle-node bifurcation occurs for the following system:

$$\dot{x} = r + x - \ln(1+x).$$

Solution:

Let $f(x) = x - \ln(1+x)$ and $g(r,x) = r + x - \ln(1+x)$. Clearly, f is a concave up function with a global minimum x_c satisfying $1 - \frac{1}{1+x_c} = 0 \Rightarrow x_c = 0$. Therefore, three representive sketches of the curve $g(r,x)$ are given by:





The saddle-node bifurcation occurs at $r=0$

#3.2.3

Show that a transcritical bifurcation occurs for the following system:

$$\dot{x} = x - rx(1-x).$$

Solution:

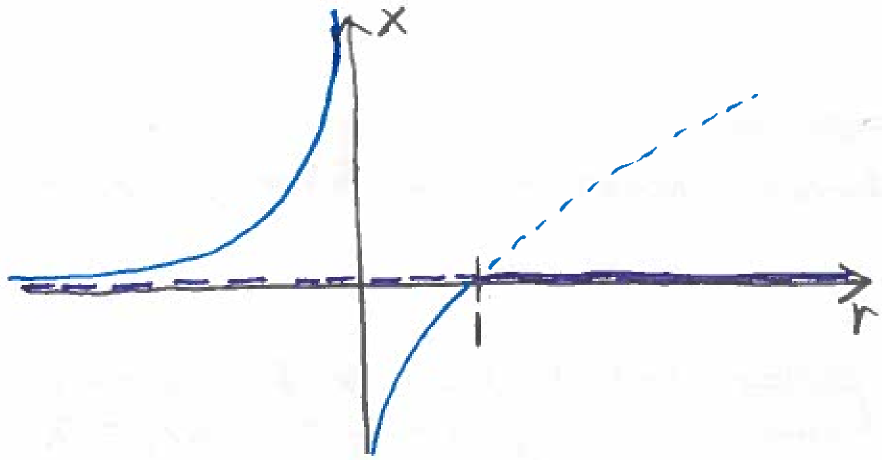
Solving for fixed points we have that:

$x^* = 0, \ \frac{r-1}{r}.$

To determine stability let $f(x) = x - rx(1-x)$ and calculate:

$$f'(0) = 1 - r + 2x\big|_{x=0} = 1 - r$$

Therefore, $x^* = 0$ is stable if and only if $r > 1$ and consequently $x^* = \frac{r-1}{r}$ is stable if and only if $r < 1$. Therefore, a transcritical bifurcation occurs at $r=1$ and we have the following bifurcation diagram:



#3.3.1

Analyze the following system:

$$\dot{n} = GnN - kn,$$
$$\dot{N} = -GnN - fN + p,$$

assuming an adiabatic approximation $\dot{N} \approx 0$.

<u>Solution:</u>

If $\dot{N} \approx 0$ then

$$N \approx \frac{p}{Gn+f}.$$

Consequently,

$$\dot{n} \approx n\left(\frac{Gp}{Gn+f} - k\right) = f(n).$$

which has fixed points at

$$n^* = 0, \frac{Gp-kf}{kG}.$$

Now since $\lim_{n\to\infty} f(n) = -\infty$ it follows that that $n^* = \frac{Gp-kf}{kG}$ is stable if and only if $p > kf/G$. Therefore, $p = kf/G$ is a transcritical bifurcation.

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#3.3.2

Analyze the following system

$$\dot{E} = K(P-E)$$
$$\dot{P} = \gamma_1(ED-P)$$
$$\dot{D} = \gamma_2(\lambda+1-D-\lambda EP)$$

Assuming $\dot{P}\approx 0$, $\dot{D}\approx 0$.

Solution:

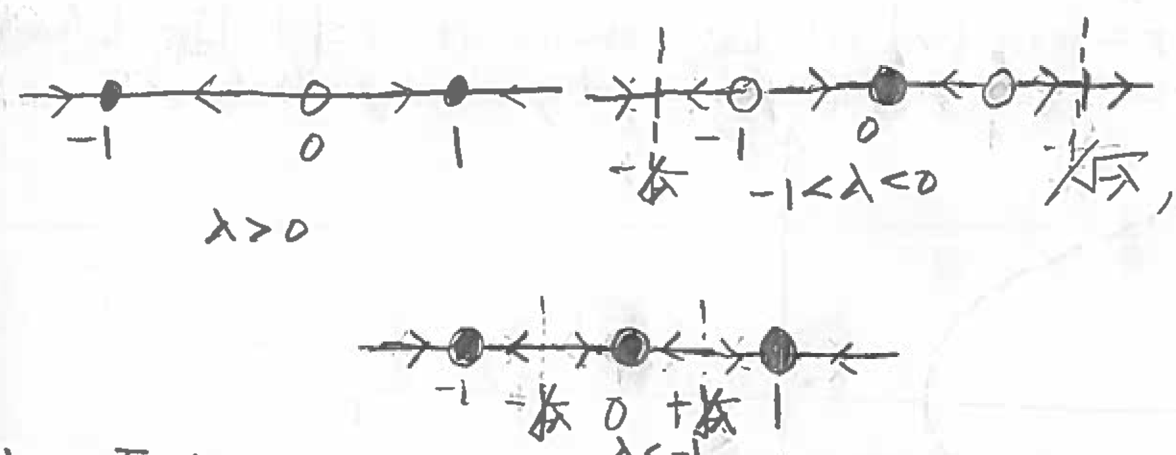
Assuming $\dot{P}\approx 0$ and $\dot{D}\approx 0$ we have

$$D = \lambda+1-\lambda EP \Rightarrow E\lambda + E - \lambda E^2P - P = 0.$$

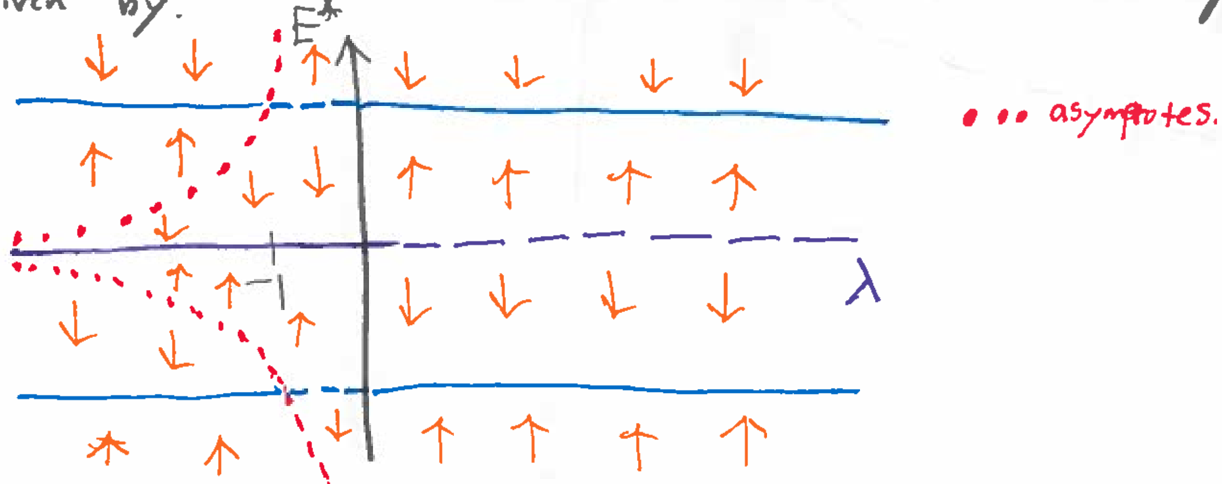
Therefore,

$$P = \frac{E\lambda+E}{1+\lambda E^2} \Rightarrow \dot{E} = \frac{E\lambda+E-(1+\lambda E^2)E}{(1+\lambda E^2)} = \frac{E\lambda(1-E^2)}{(1+\lambda E^2)}.$$

For $\lambda\neq 0$ the fixed points are given by $E^* = 0, \pm 1$. If $\lambda<0$ the equation for $\dot{E}$ is not smooth and there are asymptotes at $E=\pm 1/\sqrt{-\lambda}$. These asymptotes are similar to fixed points in that the flow changes sign across them. Below are some representative phase portraits:



Where I have used | to indicate an asympte. The bifurcation diagram is thus given by:



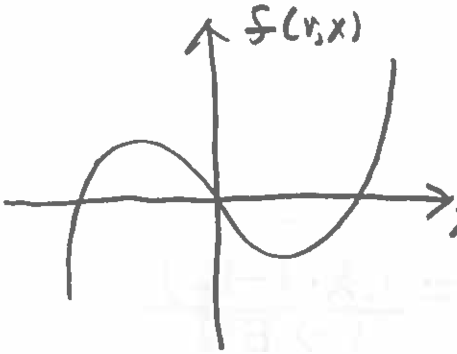
#3.4.9.

Analyze the following system:

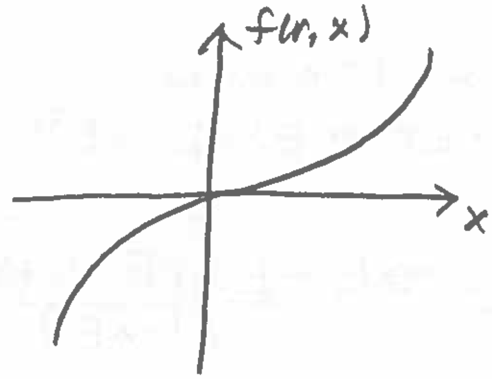
$$\dot{x} = x + \tanh(rx).$$

Solution:

Let $f(r,x) = x + \tanh(rx)$.



$r<0$



$r>0$

Clearly a pitchfork bifurcation occurs. Now,

$$\frac{\partial f}{\partial x} = 1 + r\,\mathrm{sech}^2(rx)$$

hence $f(r,x)$ only has critical points in x if and only if $r \le -1$, since the equation $r = -\cosh^2(rx)$ only has solutions if $r \le -1$. The bifurcation point is thus $r=-1$ and the bifurcation diagram is given by: