Homework #8

#8.1.5

At any zero eigenvalue bifurcation in two dimensions, the null-clines always intersect tangentially.

Solution:

True. Consider the system

$$\dot{x} = f(x, y, \nu)$$
$$\dot{y} = g(x, y, \nu)$$

and assume a zero-eigenvalue bifurcation occurs at ν^* with fixed point (x^*, y^*). Therefore,

$$\det(J(x^*, y^*)) = f_x g_y - f_y g_x = 0$$
$$\Rightarrow \nabla f \cdot (g_y, -g_x) = 0$$

Therefore, ∇f and $(g_y, -g_x)$ are orthogonal which implies $\nabla f, \nabla g$ are parallel. Consequently, since $\nabla f, \nabla g$ are outward normals to the contours of f and g it follows that null-clines intersect tangentially.

#8.1.6

Consider the system

$$\dot{x} = y - 2x$$
$$\dot{y} = \nu + x^2 - y.$$

Solution:

The null-clines are given by:

N1: $y = 2x$ $(\dot{x} = 0)$

N2: $y = x^2 + \nu$ $(\dot{y} = 0)$.

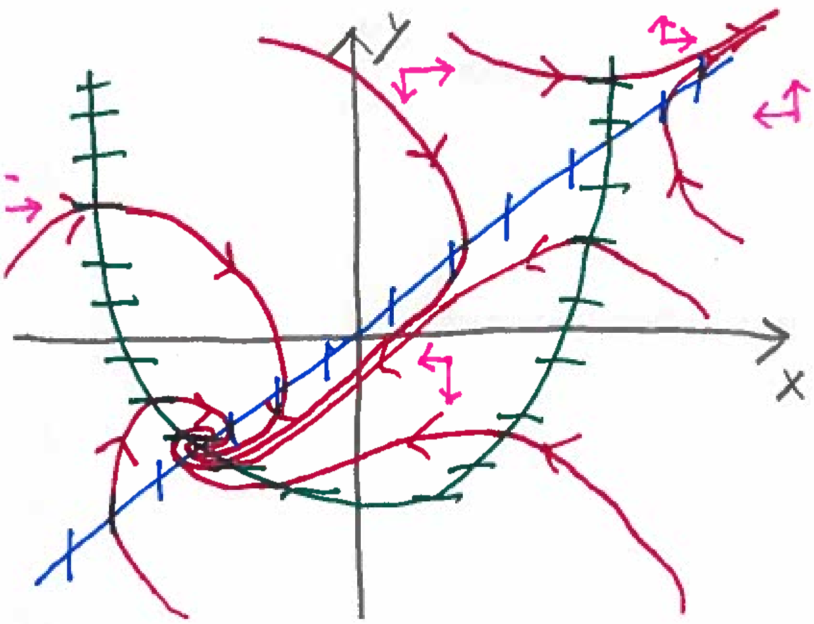
The x-coordinate of the fix point thus satisfies:

$$x^2 - 2x + \nu = 0.$$

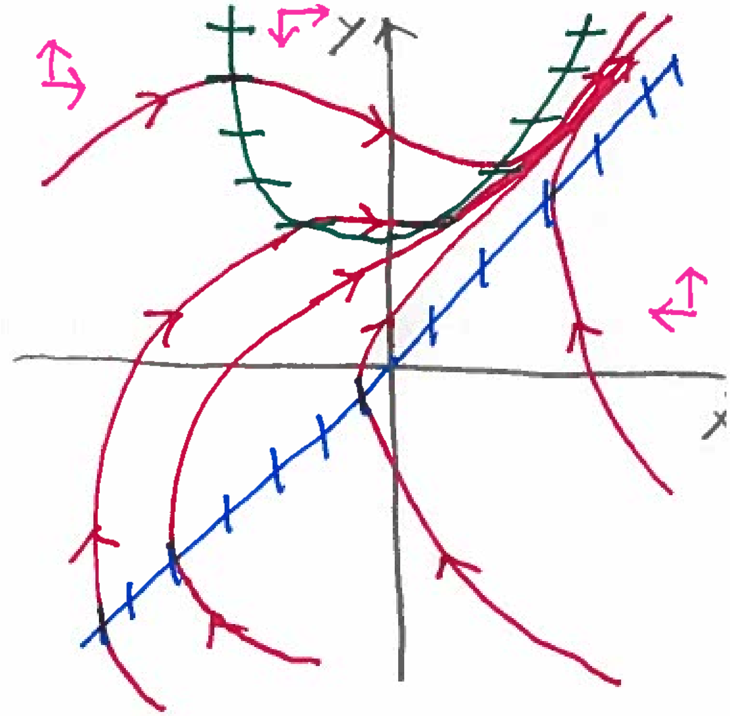
The discriminant of this quadratic is given by:

$$D = \sqrt{4 - 4\nu}.$$

Consequently, a saddle node bifurcation occurs at $\nu = 1$.



$\nu < 1$



$\nu > 1$

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#8.1.11

Show that a saddle-node bifurcation occurs for the system

$$\dot{u} = a(1-u) - uv^2$$
$$\dot{v} = uv^2 - (a+k)v$$

at $k = -a \pm \frac{1}{2}\sqrt{a}$.

Solution:

The $\dot{v}$ null-clines are given by:

$$v = 0, \quad v = \frac{a+k}{u}.$$

Consequently, the u coordinate of the fixed points satisfy:

$$u = 1, \quad a(1-u) - \frac{(a+k)^2}{u} = 0.$$

$$\Rightarrow u = 1, \quad -au^2 + au - (a+k)^2 = 0.$$

The discreminant of the quadratic polynomial is given by:

$$D = \sqrt{a^2 - 4a(a+k)^2}$$

Consequently, a bifurcation occurs when $D = 0$

$$\Rightarrow k = -a \pm \frac{1}{2}\sqrt{a}.$$

$\blacksquare$

8.1.13

Analyze the following system:

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p.$$

Solution:

Let

$$x = \frac{n}{\alpha},\ y = \frac{N}{\beta},\ \tau = \gamma t$$

$$\Rightarrow \alpha\gamma\dot{x} = G\alpha\beta xy - \alpha kx$$
$$\beta\gamma\dot{y} = -G\alpha\beta xy - f\beta y + p$$

$$\Rightarrow \dot{x} = \frac{G\beta xy}{\gamma} - \frac{k}{\gamma}$$
$$\dot{y} = -\frac{G\alpha xy}{\gamma} - \frac{f}{\gamma}y + \frac{p}{\gamma}$$

Let,

$\gamma = f$ (decay rate)

$$\alpha = \frac{\gamma}{G} = \frac{f}{G}\ \left(\frac{\text{decay rate}}{\text{gain rate}}\right)$$

$$\beta = \frac{\gamma}{G} = \frac{f}{G}\ \left(\frac{\text{decay rate}}{\text{gain rate}}\right)$$

$$\delta = \frac{k}{\gamma} = \frac{k}{f}\ \left(\frac{\text{decay rate}}{\text{decay rate}}\right)$$

$$p = \frac{p}{\gamma} = \frac{p}{f}\ \left(\frac{\text{pump rate}}{\text{decay rate}}\right)$$

$$\Rightarrow \dot{x} = xy - \delta x$$
$$\dot{y} = -xy - y + p.$$

The null-clines are given by:

N1: $x = 0$ $(\dot{x} = 0)$

N2: $y = \delta$ $(\dot{x} = 0)$

N3: $y = \dfrac{p}{1+x}$ $(\dot{y} = 0)$

The fixed points are thus given by:

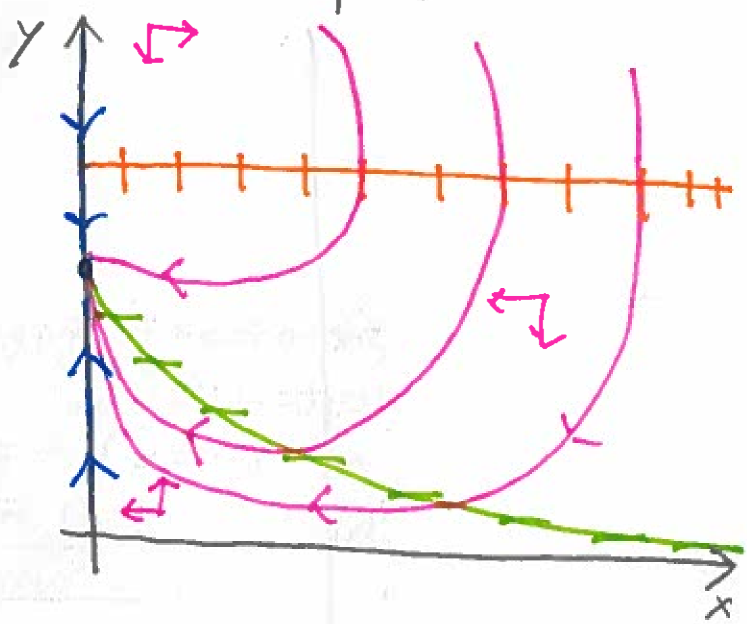
$$(0,p),\ \left(\frac{p-\delta}{\delta},\ 0\right).$$

Consequently, we have a transcritical bifurcation at $p=\delta$.

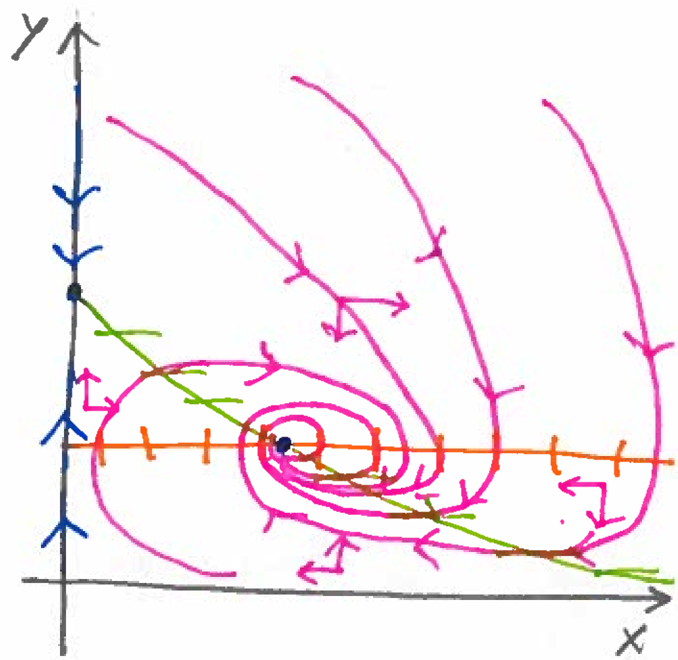
Now,

$$J(0,p)=\begin{pmatrix} p-\delta & 0 \\ -p & -1 \end{pmatrix}$$

Which has eigenvalues $\lambda_1=-1$, $\lambda_2=p-\delta$ which implies $(0,p)$ is stable if $p<\delta$.



$p<\delta$



$p>\delta$

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#8.6.1

Analyze the following system:

$$\dot{\theta}_1 = w_1 + \sin\theta_1 \cos\theta_2$$
$$\dot{\theta}_2 = w_2 + \sin\theta_2 \cos\theta_1$$

Solution:

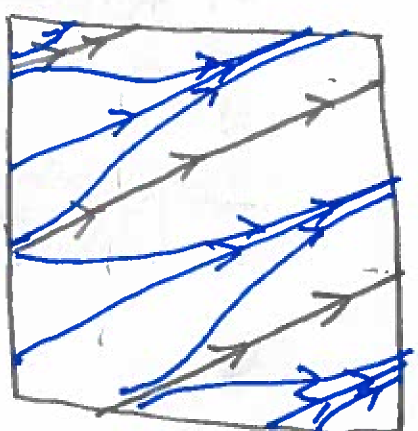
First lets determine if fixed points exist. At a fixed point we have

$$w_1 + \sin\theta_1 \cos\theta_2 = w_2 + \sin\theta_2 \cos\theta_1$$
$$\Rightarrow \sin(\theta_1 - \theta_2) = w_2 - w_1.$$

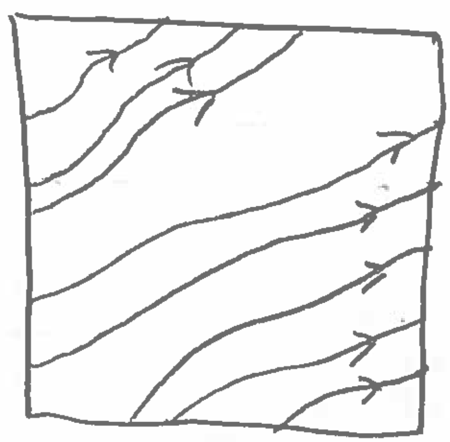
Let $\phi = \theta_1 - \theta_2$. Then,

$$\dot{\phi} = w + \sin(\phi),$$

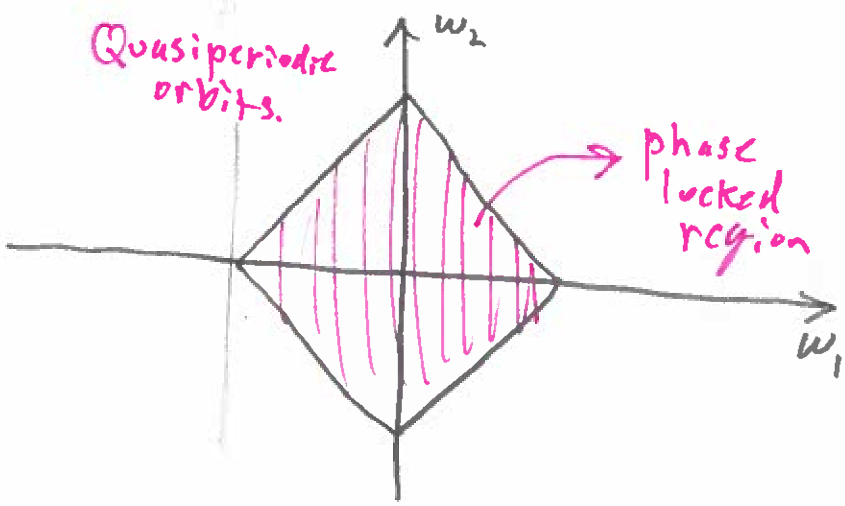
where, $w = w_1 - w_2$. Consequently, if $|w| < 1$ the system will become phase locked.



$|w| < 1$



$w > 1$



#8.6.9

Analyze the following frog model:

$$\dot{\theta}_i = w + \sum_{\substack{k=j \\ j\neq i}}^{n} H(\theta_j - \theta_i)$$

where $w>0$ is constant, H is 2π periodic.

Solution:

For $n=2$, let $\phi = \theta_1 - \theta_2$.

$$\Rightarrow \dot{\phi} = -2H(\phi). \qquad (1)$$

For $n=3$, let $\phi = \theta_1 - \theta_2$, $\psi = \theta_2 - \theta_3$

$$\Rightarrow \dot{\phi} = -2H(\phi) + H(\theta_3 - \theta_1) - H(\psi)$$
$$= -2H(\phi) + H(\theta_3 - \theta_2 + \theta_2 - \theta_1) - H(\psi)$$
$$= -2H(\phi) - H(\phi+\psi) + H(\psi).$$

$$\dot{\psi} = -2H(\psi) - H(\theta_1 - \theta_3) + H(\phi)$$

$$\Rightarrow \begin{aligned} \dot{\phi} &= -2H(\phi) - H(\phi+\psi) + H(\psi) \\ \dot{\psi} &= -2H(\psi) - H(\phi+\psi) + H(\phi) \end{aligned} \qquad (2).$$

Now, if $H(x) = a\sin(x)$ it follows that (1) becomes

$$\dot{\phi} = -2a\sin(\phi).$$

Therefore, if $a<0$ the two frogs will phase seperate by π.

However, for three frogs:

$$\dot{\phi} = -2a\sin(\phi) - a\sin(\phi+\psi) + a\sin(\psi)$$
$$\dot{\psi} = -2a\sin(\psi) - a\sin(\phi+\psi) + a\sin(\phi)$$

Now,

$$-2a\sin(2\pi/3) - a\sin\left(\tfrac{4\pi}{3}\right) + a\sin\left(\tfrac{2\pi}{3}\right) = -\sqrt{3}a + \frac{\sqrt{3}}{2}a + \frac{\sqrt{3}}{2}a = 0$$

Consequently, $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ is a fixed point for this system.

The Jacobian at this point is given by:

$$J\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \begin{pmatrix} +\frac{3}{2}a & 0 \\ 0 & +\frac{3}{2}a \end{pmatrix}$$

which implies $(2\pi/3, 2\pi/3)$ is stable if $a<0$. However, if $\phi=0$ $\psi=\pi$ then,

$$J(0,\pi) = \begin{pmatrix} -a & 0 \\ 3a & 2a \end{pmatrix}$$

Which implies $(0,\pi)$ is a saddle point. This contradicts what is experimentally observed.

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