Period Doubling Route to Chaos and Universality

Example

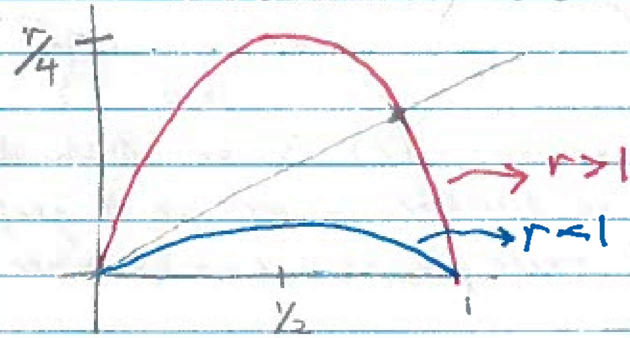
$x_{n+1} = r x_n (1 - x_n), \quad 0 \le x_n \le 1, \; 0 \le r \le 4$

Fixed points are given by $x^*_{1,2} = 0, 1 - \frac{1}{r}$.

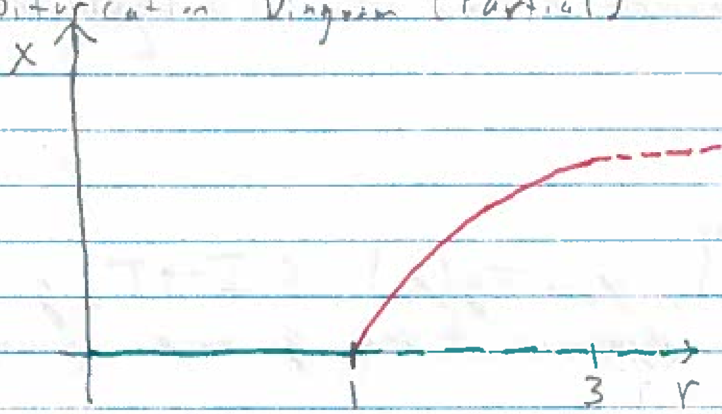
Let $f(x) = r x (1-x)$ Then

$$f'(x) = r - 2rx$$

$$\Rightarrow f'(0) = r \qquad f'\left(1 - \frac{1}{r}\right) = 2 - r.$$



Bifurcation Diagram (Partial)



What about period two orbits? $f(f(x)) = x$ (quartic poly)

$$x^*_{3,4} = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

Two periodic orbits exist if

$$(r-3)(r+1) > 0 \Rightarrow r > 3 \text{ or } r < -1$$

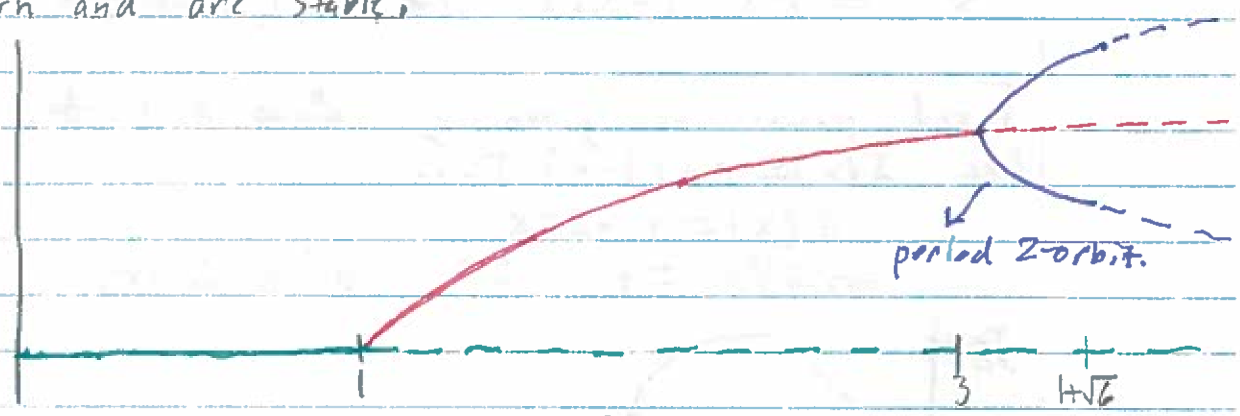
To analyze stability we evaluate the derivative

$$\frac{d}{dx}\left(f^2(x_3^*)\right) = \left(-1 + \sqrt{(r-3)(r+1)}\right)\left(-1 - \sqrt{(r-3)(r+1)}\right) = 4 + 2r - r^2$$

Solving

$$|4+2r-r^2|<1 \Rightarrow 3<r<1+\sqrt{6}.$$

$\Rightarrow$ At the bifurcation point the period 2 cycles are born and are stable;



What about period 3 orbits? $f^3(x)$ is an 8-th degree polynomial. This is hard to determine so we turn to graphical analysis. At $r_\infty = 3.57$ there is an infinite number of periodic orbits $\Longrightarrow$ chaos

$\downarrow$

The set of all unstable periodic orbits is a topological Cantor set.

Conjugacy:

When do two maps

$$x_{n+1} = f(x_n), \quad y_{n+1} = g(y_n), \quad f: I \to I, \quad g: J \to J$$

generate the "same" dynamics. f and g are conjugate if $\exists h$ such that

$$f = h \circ g \circ h^{-1}$$

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "h"'] & I \\ J \arrow[r, "g"] & J \arrow[u, "h^{-1}"'] \end{tikzcd}$$

h – 1-1 and onto

$y_{n+1} = y_n^2 + c$, $x_{n+1} = r x_n(1-x_n)$

Let $x_n = a y_n + b$

$$\Rightarrow a y_{n+1} + b = r(a y_n + b) - r(a y_n + b)^2$$
$$= r a y_n + b - r(a^2 y_n^2 + 2ab y_n + b^2)$$

Set

$ra - 2abr = 0$ and $ra = -1$

$\Rightarrow b = 1/2$ $\Rightarrow a = -1/r$

$$\Rightarrow -\frac{1}{r} y_{n+1} + \frac{1}{2} = \frac{r}{2} - \frac{1}{r} y_n^2 - r b^2$$

$$\Rightarrow y_{n+1} = y_n^2 + \underbrace{\frac{r}{2} - \frac{r^2}{2} + r b^2}_{c}$$

We can transform between c and r using this formula

The dynamics will be the same.