

Homework 2

Analysis

Due: January 29, 2018

1. Metric Spaces:

- (a) Prove that (\mathbb{R}, d) is a metric space with metric $d(x, y) = |x - y|$.
- (b) Prove that (\mathbb{R}^n, d_1) is a metric space with metric

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n |y_i - x_i|.$$

- (c) Prove that (\mathbb{R}^n, d_∞) is a metric space with metric

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i \in \{1, \dots, n\}} |y_i - x_i|.$$

- (d) Let X be the set of n -letter words in a k character alphabet $A = \{a_1, a_2, \dots, a_k\}$, meaning that $X = \{(x_1, x_2, \dots, x_n) : x_i \in A\}$. We define the distance between two words $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ to be the number of places in which the words have different letters. That is,

$$d(x, y) = \#\{i : x_i \neq y_i\}.$$

Prove that (X, d) is a metric space.

2. Discrete Metric:

- (a) Prove that (\mathbb{R}^n, d) is a metric space with metric

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

This metric space is called the **discrete metric**.

- (b) Show that in this metric space topology every subset of \mathbb{R} is open.
- (c) Show that in this metric space topology a sequence x_n converges to $x \in \mathbb{R}^n$ if and only if there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n = x$.

3. Building Metric Spaces: (X, d_X) and (Y, d_Y) are metric spaces.

- (a) If $U \subseteq X$, show that (U, d_U) is a metric space where $d_U(u_1, u_2) = d_X(u_1, u_2)$ for all $u_1, u_2 \in U$.
- (b) Show that the Cartesian product $X \times Y$ is a metric space with the metric

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

4. Normalizing a Metric Space:

- (a) Recall that a continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ is concave if for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$

$$(1 - \alpha)f(x) + \alpha f(y) \leq f((1 - \alpha)x + \alpha y).$$

What does this inequality mean geometrically?

- (b) Prove that if $f(0) \geq 0$ and f is concave then for all $t \in [0, 1]$:

$$f(tx) \geq tf(x).$$

- (c) Prove that if $f(0) \geq 0$ and f is concave then for all $a, b \in \mathbb{R}$:

$$f(a + b) \leq f(a) + f(b).$$

- (d) Show that if (X, d) is a metric space then (X, ρ) is a metric space, where

$$\rho(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}.$$

5. **Convergence:** Let x_n be a sequence in \mathbb{R} . Recall that the **limit superior** of x_n , or $\limsup x_n$, and **limit inferior** of x_n , or \liminf , are defined by

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &= \sup\{x_k : k \geq n\}, \\ \liminf_{n \rightarrow \infty} x_n &= \inf\{x_k : k \geq n\}. \end{aligned}$$

- (a) Prove that if x_n is bounded then $\limsup_{n \rightarrow \infty} x_n$ and $\liminf_{n \rightarrow \infty} x_n$ exist in the sense that they are both finite numbers.
- (b) Prove that

$$\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n.$$

- (c) Let x_n be a sequence of real numbers. A point $c \in \mathbb{R} \cup \{\pm\infty\}$ is called a **cluster point** of x_n if there is a convergent subsequence of x_n with limit c . Let C denote the set of cluster points of x_n . Prove that

$$\limsup_{n \rightarrow \infty} x_n = \max C \text{ and } \liminf_{n \rightarrow \infty} x_n = \min C.$$

- (d) If x_n is a sequence of real numbers such that

$$\lim_{n \rightarrow \infty} x_n = x,$$

and $a_n \leq x_n \leq b_n$, prove that

$$\limsup_{n \rightarrow \infty} a_n \leq x \leq \liminf_{n \rightarrow \infty} b_n.$$