MST 352/652
Spring 2019
Exam 1
02/14/19

Name (Print): Key

This exam contains 9 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Undergraduate Problems:** Questions labeled as **"Undergraduate problems"** will only count for undergraduate students. Graduate students do not have to complete these problems.
- **Undergraduate Problems:** Questions labeled as **"Graduate problems"** must be completed by the graduate students to receive credit. Undergraduate students can complete these problems for extra credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	10	
2	10	
3	5	
4	15	
5	15	
6	10	
7	15	
8	20	
9	20	
Total:	120	

Do not write in the table to the right.

1. (10 points) (**Short Answer**) Determine if the following statement is correct (**C**) or incorrect (**I**). Just circle **C** or **I**. No need to show any work. In order for a statement to be correct it must be true in all cases.

C **I** Consider the initial value problem

$$\begin{cases} \dfrac{dx}{dt} = f(x(t)) \\ x(0) = x_0. \end{cases}$$

If f is differentiable and f' is continuous then solutions to this initial value problem are unique and exist for all t.

C **I** Suppose L is a linear differential operator. If u solves $L[u] = f$ and v solves $L[v] = 0$ then $w = u + v$ also solves $L[w] = f$.

C **I** Suppose L and M are linear differential operators and let $N = L + M$. If u solves $L[u] = f$ and v solves $M[v] = g$ then $w = u + v$ solves $N[w] = f + g$.

C **I** Suppose f, g are smooth functions that can be represented by the following Fourier series expansions:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) + \sum_{n=1}^{\infty} d_n \sin(nx).$$

The Fourier series expansion of $f + g$ is given by

$$f(x) + g(x) \sim \frac{a_0 + c_0}{2} + \sum_{n=1}^{\infty} (a_n + c_n) \cos(nx) + \sum_{n=1}^{\infty} (b_n + d_n) \sin(nx).$$

C **I** Suppose f, g are smooth functions that can be represented by the following Fourier series expansions:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$g(x) \sim \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx) + \sum_{n=1}^{\infty} d_n \sin(nx).$$

The Fourier series expansion of $f \cdot g$ is given by

$$f(x)g(x) \sim \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty} a_n c_n \cos(nx) + \sum_{n=1}^{\infty} b_n d_n \sin(nx).$$

2. (10 points) Let $D > 0$.

   (a) (5 points) Assuming $a < 0$ is a constant, for what values of b is $u(t,x) = e^{at}\sin(bx)$ a solution to the equation

$$u_t = Du_{xx}?$$

$$U_t = ae^{at}\sin(bx)$$
$$U_{xx} = -b^2 e^{at}\sin(bx)$$
$$\Rightarrow a = -b^2 D$$

   (b) (5 points) For what values of a and b is $u(t,x) = e^{at}\sin(bx)$ a solution to the following boundary value problem:

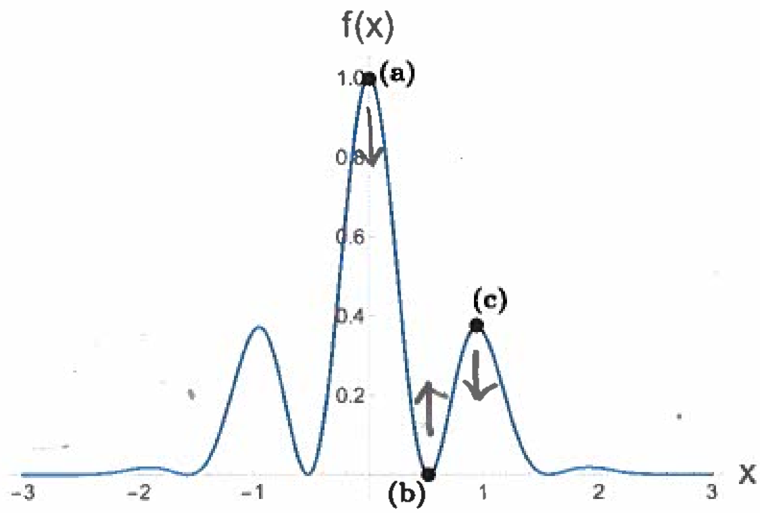
$$u_t = Du_{xx}$$
$$u(t,0) = 0$$
$$u(t,\pi) = 0.$$

$a = -b^2 D$ and $b = n\pi$, $n \in \mathbb{N}$.

3. (5 points) Suppose $u(t, x)$ solves the following initial boundary value problem:

$$u_t = u_{xx},$$
$$u(0, x) = f(x),$$
$$u_x(t, -3) = u_x(t, 3) = 0,$$

where $f(x)$ is plotted below.



(a) (5 points) **Undergraduate Problem, Short Answer:** At points **(a)**, **(b)**, and **(c)** indicate on the figure whether $u(t, x)$ is increasing or decreasing in time at $t = 0$.

(b) (5 points) **Graduate Problem:** Prove that the total heat defined by

$$H(t) = \int_{-3}^{3} u(t, x)\, dx$$

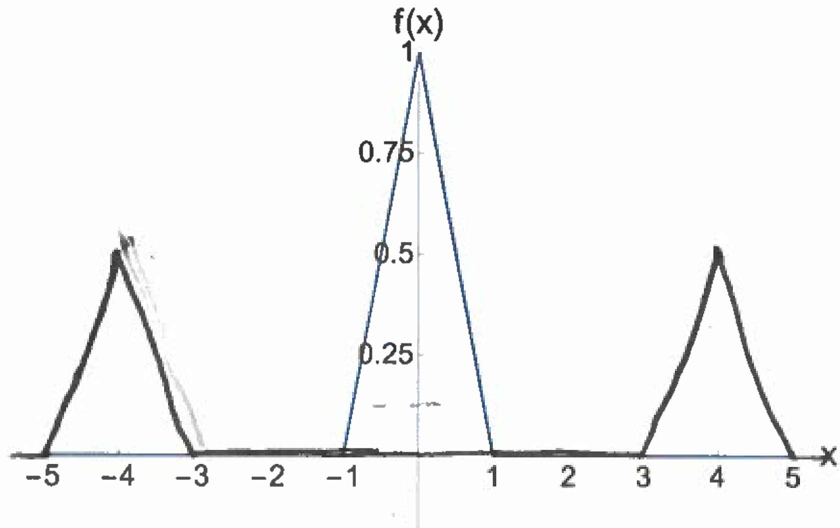
is conserved in time.

$$\frac{dH}{dt} = \int_{-3}^{3} u_t(t,x)\,dx$$
$$= \int_{-3}^{3} u_{xx}(t,x)\,dx$$
$$= u_x\Big|_{-3}^{3}$$
$$= 0.$$

4. (15 points) Suppose $u(t,x)$ solves the following initial value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx},\ x \in \mathbb{R},\ t > 0 \\ u(0,x) &= f(x), \\ u_t(0,x) &= 0, \end{aligned}$$

where $f(x)$ is plotted below.



(a) (5 points) **Short Answer:** On the same set of axes, carefully sketch a graph of $u(1,x)$. Be sure to carefully sketch the location of any local maximum, minimum and zeros.

(b) (5 points) **Short Answer:** What is $\lim_{t\to\infty} u(t,x)$ for all $x \in \mathbb{R}$?

$$\lim_{t\to\infty} u(t,x) = 0$$

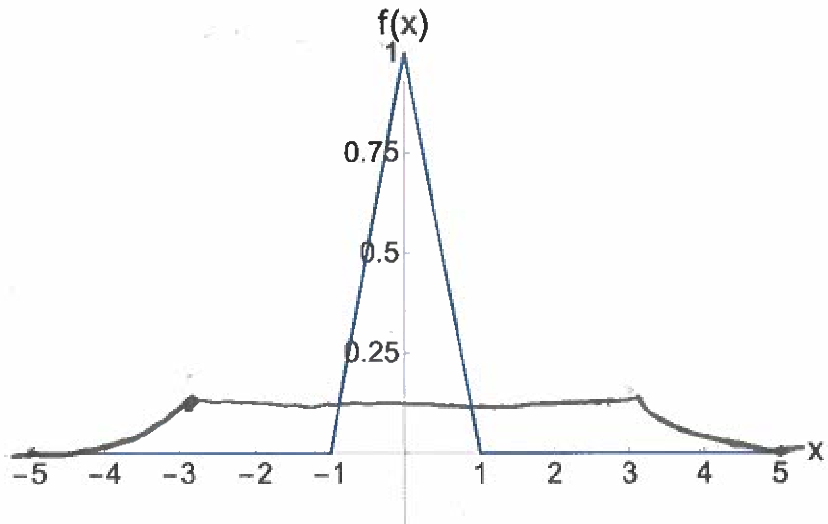
(c) (5 points) **Short Answer:** What is $\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)dx$?

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)\,dx = 1$$

5. (15 points) Suppose $u(t,x)$ solves the following initial value problem:

$$\begin{aligned} u_{tt} &= 4u_{xx},\ x \in \mathbb{R},\ t > 0 \\ u(0,x) &= 0, \\ u_t(0,x) &= f(x), \end{aligned}$$

where $f(x)$ is plotted below.



(a) (5 points) **Undergraduate Problem, Short Answer:** Write down the generic form of the solution to this problem. **You do not have to evaluate the integrals. I am just looking for the formula.**

$$u(t,x) = \frac{1}{8}\int_{x-4t}^{x+4t} f(s)\,ds.$$

(b) (5 points) **Graduate Problem, Short Answer:** On the same set of axes, carefully sketch a graph of $u(1/2, x)$. Be sure to carefully sketch the location of any local maximum, minimum and zeros.

(c) (5 points) **Short Answer:** What is $\lim_{t\to\infty} u(t,x)$ for all $x \in \mathbb{R}$?

$$\lim_{t\to\infty} u(t,x) = \frac{1}{8}$$

(d) (5 points) **Short Answer:** What is $\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)dx$?

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} u(t,x)\,dx = \infty.$$

6. (10 points) Solve the following initial value problem:

$$u_t + 2u_x = \cos(x)$$
$$u(0,x) = \cos(x).$$

Let $z = x - 2t$, $\tau = t$

$\Rightarrow u_\tau = \cos(z + 2\tau)$

$\Rightarrow u(\tau, z) = \frac{1}{2}\sin(z+2\tau) + f(z)$

$\Rightarrow u(t, x) = \frac{1}{2}\sin(x) + f(x-2t)$

$\Rightarrow u(0,x) = \frac{1}{2}\sin(x) + f(x) = \cos(x)$

$\Rightarrow f(x) = \cos(x) - \frac{1}{2}\sin(x)$

$\Rightarrow u(t,x) = \frac{1}{2}\sin(x) + \cos(x-2t) - \frac{1}{2}\sin(x-2t).$

7. (15 points) Find all separable solutions to the following partial differential equation:

$$u_{tt} = u_{xx} + u.$$

Let $u(t,x) = T\,X$

$\Rightarrow T''X = TX'' + TX$

$\Rightarrow \frac{T''}{T} = \frac{X'' + X}{X} = \lambda$

$\Rightarrow T'' = \lambda T, \quad X'' + X = \lambda X$

$\Rightarrow T = (Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t})$

$X = (Ce^{\sqrt{\lambda-1}x} + De^{-\sqrt{\lambda-1}x})$

8. (20 points) **Undergraduate Problem:** Consider the following initial value problem.

$$u_t - cxu_x = 0$$
$$u(0,x) = e^{-x^2},$$

where $c > 0$ is a constant.

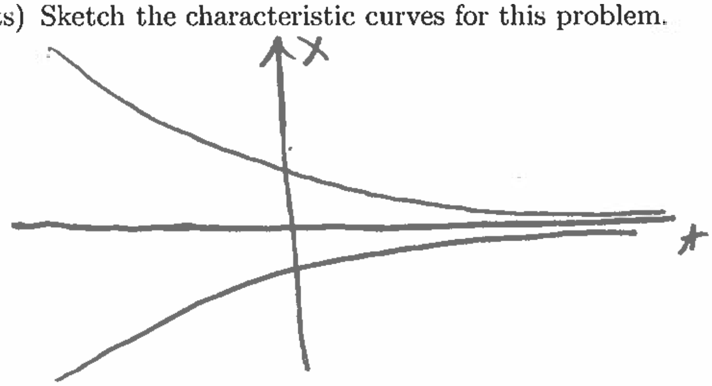
(a) (5 points) **Short Answer:** Write down the differential equation satisfied by the characteristic curves for this problem.

$$\frac{dx}{dt} = -cx$$

(b) (5 points) Solve the differential equation satisfied by the characteristic curves.

$$x(t) = De^{-cx}$$

(c) (5 points) Sketch the characteristic curves for this problem.



(d) (5 points) For all $x \in \mathbb{R}$, calculate the following limit:

$$\lim_{t\to\infty} u(t,x).$$

$$\lim_{t\to\infty} u(t,x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

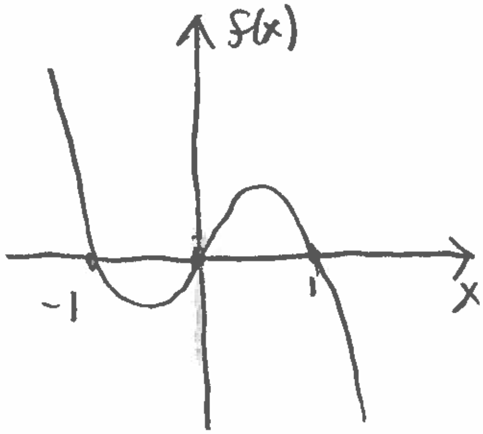
9. (20 points) **Graduate Problem:** Consider the following initial value problem.

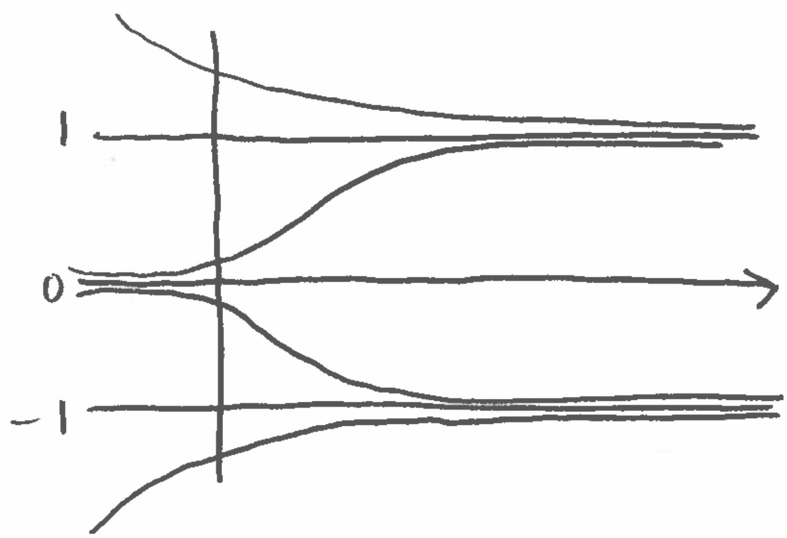
$$u_t - x(x-1)(x+1)u_x = 0$$
$$u(0,x) = e^{-x^2}.$$

(a) (5 points) **Short Answer:** Write down the differential equation satisfied by the characteristic curves for this problem.

$$\frac{dx}{dt} = -x(x-1)(x+1) = f(x)$$

(b) (5 points) Sketch the characteristic curves for this problem.





(c) (10 points) For all $x \in \mathbb{R}$, calculate the following limit:

$$\lim_{t\to\infty} u(t,x).$$

$$\lim_{t\to\infty} u(t,x) = \begin{cases} e^{-1}, & x=-1,1 \\ 1, & -1<x<1 \\ 0 & |x|>1. \end{cases}$$