

MST 352/652

Homework #1

Due Date: January 22, 2019

1 Problems for everyone

1. pg. 8, #1.5-8, #1.10, #1.11
2. pg. 13, #1.19, #1.20, #1.22, #1.24-1.26
3. For what values of a and b is the function $u(x, t) = e^{at} \sin(bx)$ a solution to the heat equation:

$$u_t = ku_{xx}.$$

2 Graduate Problems

Undergraduate students can complete these exercises for extra credit The goal of these exercises is to introduce the notion of **dispersion relationships** and **linear superposition**. Namely, you will learn how solutions to linear partial differential equations can be built up out of a **linear superposition of plane wave solutions**.

1. (**Dispersion Relationships** Linear, homogeneous partial differential equations with constant coefficients admit complex solutions of the form

$$u(x, t) = Ae^{i(kx - \omega t)}$$

which are called **plane waves**. The real and imaginary parts of this complex function give real solutions. The constant A is the **amplitude**, k is the **wave number**, and ω is the **temporal (angular) frequency**. When the plane wave form is substituted into the partial differential equation there results a dispersion relationship of the form

$$\omega = \omega(k)$$

which states how the frequency depends upon the wave number in order for the plane wave to be a solution. For the following partial differential equations find the dispersion relationship and determine the resulting plane wave:

- (a) $u_t = u_{xx}$ (heat equation)
- (b) $u_{tt} = u_{xx}$ (wave equation)
- (c) $u_t = iu_{xx}$ (Schrödinger's equation)

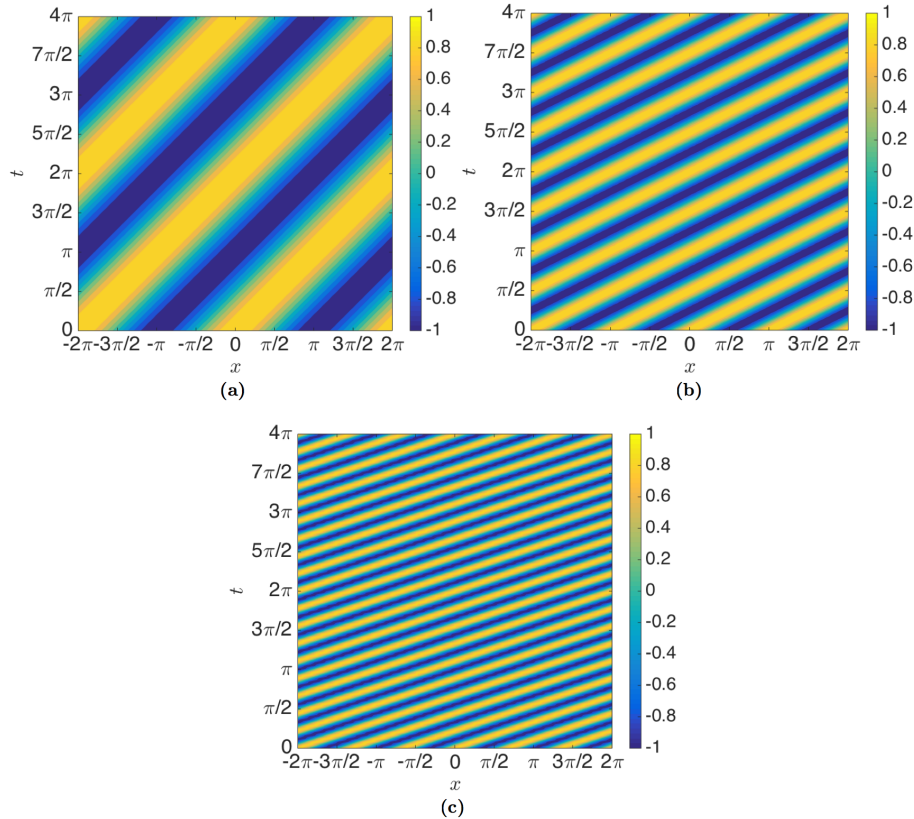
2. **(Superposition of Solutions)** For the dispersion relationships determined above for the heat and Schrödinger's equation, show that

$$u(x, t) = \int_{-\infty}^{\infty} f(k)e^{i(kx - \omega(k)t)} dk$$

is also a solution to the partial differential equation for any function $f(k)$ that is bounded and continuous on $[0, \infty)$. This exercise illustrates that more complicated solutions to partial differential equations can be built up from plane waves.

3. **Backwards** Figures (a-c) below are contour plots for the *real component* of plane wave solutions $u(x, t) = e^{i(kx - \omega(k)t)}$ to an unknown linear partial differential equation.

- (a) For each figure determine the value of k and ω . **Hint:** Think about how k and ω are related to spatial wavelength and temporal period.
 (b) Make a conjecture of the dispersion relationship $\omega(k)$.
 (c) From this dispersion relationship determine the partial differential equation these plane wave solutions satisfy.



Historical note: This particular partial differential equation was derived by physicists in this exact manner. Experiments could measure a relationship between spatial wavelength and temporal frequency which when combined gave a dispersion relationship and equivalently a partial differential equation!