MST 352/652
Homework #9

Due Date: April 09, 2019

1 Problems for everyone

1. pg. 170-172, #4.3.25, #4.3.26, #4.3.28, #4.3.31, #4.3.34, #4.3.36.
2. Consider the following boundary value problem on a quarter wedge of radius R:

$$\begin{aligned}
&\Delta u = 0, r < R,\ 0 \le \theta \le \pi/2, \\
&u(R,\theta) = \sin(2\theta), \\
&u(r,0) = u(r,\pi/2) = 0.
\end{aligned}$$

   (a) Solve this boundary value problem.

   (b) Sketch a contour plot of your solution. (If you want to, you can use software to do this.)

3. Consider the following boundary value problem on an annulus:

$$\begin{aligned}
&\Delta u = 0,\ 1 < r < 2,\ 0 \le \theta \le 2\pi, \\
&u(1,\theta) = 0, \\
&u(2,\theta) = \sin^2(\theta).
\end{aligned}$$

   (a) What additional boundary conditions must be imposed to make this problem well posed?

   (b) Solve this boundary value problem.

   (c) Sketch a contour plot of your solution. (If you want to, you can use software to do this.)

4. pg. 227-228, #6.1.1, #6.1.2, #6.1.4-6.1.6, #6.1.23.

2 Graduate Problems

1. #6.1.7-6.1.10, #6.1.24.

Homework #10

#4.3.25

Solve the following boundary value problems:

a.) $\Delta u = 0,\ x^2+y^2<1,\ u = x^3,\ x^2+y^2=1,$

b.) $\Delta u = 0,\ x^2+y^2<2,\ u = \ln(x^2+y^2),\ x^2+y^2=1.$

c.) $\Delta u = 0,\ x^2+y^2<4,\ u = x^4,\ x^2+y^2=4.$

d.) $\Delta u = 0,\ x^2+y^2<1,\ \frac{\partial u}{\partial n} = x,\ x^2+y^2=1.$

Solution:

The generic solution for all of these problems is

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta).$$

a.) Since $\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$ it follows that

$$u(r,\theta) = \frac{3}{4} r\cos(x) + \frac{1}{4} r^3 \cos(3x).$$

b.) Since $\ln(1) = 0$ it follows that

$$u(r,\theta) = 0.$$

c.) Since $\cos^4(x) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$ it follows that

$$u(r,\theta) = \frac{3}{8} + \frac{1}{32} r^4 \cos(2x) + \frac{1}{256} r^4 \cos(4x).$$

d.) The boundary conditions imply

$$u(r,\theta) = A + r\cos(\theta),$$

where A is a generic constant.

#4.3.36

Solve the following boundary value problem

$$(x^2+y^2)(u_{xx}+u_{yy})+2xu_x+2yu_y=0,\ x^2+y^2<1,\ u(x,y)=1+3x,\ x^2+y^2=1$$

Solution:

Since

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

it follows that

$$r^2\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right)+2r\cos^2\theta u_r-\cos\theta\sin\theta u_\theta+2r\sin^2\theta u_r+\cos\theta\sin\theta u_\theta=0$$

$$\Rightarrow r^2u_{rr}+3ru_r+u_{\theta\theta}=0.$$

Therefore, it follows that the generic solution is of the form:

$$u(r,\theta)=a_0+\sum_{n=1}^{\infty}a_nr^{-1+\sqrt{1+n^2}}\cos(n\theta)+\sum_{n=1}^{\infty}b_nr^{-1+\sqrt{1+n^2}}\sin(n\theta).$$

Applying boundary conditions it follows that

$$u(r,\theta)=1+3r^{-1+\sqrt{2}}\cos(\theta).$$

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#2.

Solve the following boundary value problem

$$\Delta u=0,\quad r<R,\quad 0\le\theta\le\pi/2$$
$$u(R,\theta)=\sin(2\theta)$$
$$u(r,0)=u(r,\pi/2)=0.$$

a.) Solve this boundary value problem.

b.) Sketch a contour plot of your solution.

Solution:

a.) The generic form of the solution is given by

$$u(r,\theta)=\sum_{n=1}^{\infty}b_nr^{2n}\sin(2n\theta).$$

Boundary conditions imply

$$u(r,\theta)=r^2\sin(2n\theta).$$

b.) See attached.

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#2

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In[8]:= u[r_, θ_] := r^2 * Sin[2 * θ];

In[16]:= ContourPlot[{u[Sqrt[x^2 + y^2], ArcTan[y / x]]}, {x, 0, 1}, {y, 0, 1},
  RegionFunction → Function[{x, y, z}, x^2 + y^2 < 1], PlotPoints → 100,
  PlotRange → {0, 1}, Contours → 40, ColorFunction → "TemperatureMap"]
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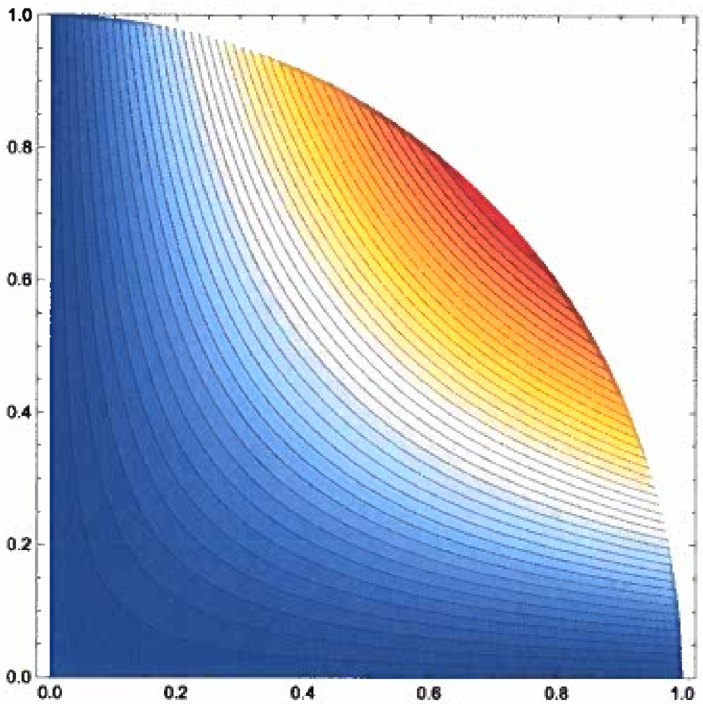
Out[16]=



#6.1.2.

Simplify the following

a.) $e^x\delta(x)$

c.) $3\delta_1(x)-3x\delta_{-1}(x)$

e.) $\cos(x)[\delta(x)+\delta(x-\pi)+\delta(x+\pi)]$

f.) $\dfrac{\delta_1(x)-\delta_2(x)}{x^2+1}$

Solution:

a.) Since $\int_{-\infty}^{\infty} e^x\delta(x)f(x)\,dx = e^0 f(0) = f(0)$ it follows that $e^x\delta(x)=\delta(x)$

c.) Since $\int_{-\infty}^{\infty}(3\delta_1(x)-3x\delta_{-1}(x))f(x)\,dx = 3f(1)+3f(-1)$ it follows that

$3\delta_1(x)-3x\delta_{-1}(x) = 3\delta_1(x)+3\delta_{-1}(x).$

e.) Since $\int_{-\infty}^{\infty}\cos(x)[\delta(x)+\delta(x-\pi)+\delta(x+\pi)]f(x)\,dx = \cos(0)f(0)+\cos(\pi)f(\pi)+\cos(-\pi)f(-\pi)$

it follows that

$$\cos(x)[\delta(x)+\delta(x-\pi)+\delta(x+\pi)] = \delta(x)-\delta(x-\pi)-\delta(x+\pi).$$

f.) Since $\int_{-\infty}^{\infty}\dfrac{\delta_1(x)-\delta_2(x)}{1+x^2}f(x)\,dx = \frac{1}{2}f(1)-\frac{1}{5}f(2)$ it follows that

$$\frac{\delta_1(x)-\delta_2(x)}{1+x^2} = \frac{1}{2}\delta_1(x)-\frac{1}{5}\delta_2(x).$$

#6.1.5

Find the first and second derivatives of

a.) $f(x)=\begin{cases} x+1, & -1<x<0, \\ 1-x, & 0<x<1, \\ 0, & \text{o.w.}\end{cases}$

b.) $k(x)=\begin{cases} |x|, & -2<x<2 \\ 0, & \text{o.w.}\end{cases}$

c.) $s(x)=\begin{cases} 1+\cos(\pi x), & -1<x<1 \\ 0, & \text{o.w.}\end{cases}$

Solution:

a.) $f'(x) = \begin{cases} 1, & -1<x<0 \\ -1, & 0<x<1 \\ 0, & \text{o.w.} \end{cases}$

$\Rightarrow f''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1)$

b.) $k'(x) = \begin{cases} -1, & -2<x<0 \\ 1, & 0<x<2 \\ 0, & \text{o.w} \end{cases} + 2\delta(x+2) - 2\delta(x-2)$

$\Rightarrow k''(x) = -\delta(x+2) + 2\delta(x) - \delta(x-2) + 2\delta'(x+2) + 2\delta'(x-2)$

c.) $s'(x) = \begin{cases} -\pi\sin(\pi x), & -1<x<1 \\ 0, & \text{o.w.} \end{cases}$

$\Rightarrow s''(x) = \begin{cases} -\pi^2\cos(\pi x), & -1<x<1 \\ 0, & \text{o.w} \end{cases} + \pi^2\delta(x+1) - \pi^2\delta(x-1).$

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#6.1.24

Write out a proof that there is no continuous function $\delta_\xi(x)$ satisfying $\int_{-\infty}^{\infty} f(y)\delta_\xi(x)dx = f(\xi)$.

proof:

Suppose there exists a continuous function $\delta_\xi(x)$ satisfying $\int_{-\infty}^{\infty} f(y)\delta_\xi(x)dx = f(\xi)$ for all $f \in C_0^\infty(\mathbb{R})$. Without loss of generality assume $\xi = 0$ and let $g \in C_0^\infty(\mathbb{R})$ satisfy $g(x) = 1$ for $-1 \le x \le 1$. Therefore,

$$\delta_\xi(0) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \delta_\xi(x)dx = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \delta_\xi(x)g(x)dx = \lim_{\Delta x \to 0} \frac{1}{\Delta x} = \infty.$$

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