MST 352/652
Homework #11

Due Date: April 16, 2019

1 Problems for everyone

1. pg. 240-241, #6.2.1, #6.2.10-6.2.14.
2. Find the Green's function for the following boundary value problem on $\mathbb{R}$:

$$u'' - u = f(x), \quad \lim_{x \to -\infty} u(x) = 0, \quad \lim_{x \to \infty} u(x) = 0.$$

2 Graduate Problems

1. Find the Green's function for the following boundary value problem:

$$\begin{aligned} u''''(x) &= f(x) \\ u(0) &= u(1) = 0 \\ u'(0) &= u'(1) = 0. \end{aligned}$$

Homework #11

#6.2.1

Let $c>0$. Find the Green's function for the boundary value problem

$$\begin{cases} -cu'' = f(x) \\ u(0)=0 \\ u'(1)=0 \end{cases}.$$

Verify that your integral formula is correct.

Solution:

The Green's function satisfies

$$-c\, G''(x,y) = \delta(x-y).$$

With the boundary conditions it follows that:

$$G(x,y) = \begin{cases} Ax, & x<y \\ B, & x>y \end{cases}$$

Continuity implies $Ay = B$ and the jump condition implies $A = 1/c$.
Therefore,

$$G(x,y) = \begin{cases} \frac{1}{c}x, & x<y \\ \frac{1}{c}y, & x>y \end{cases}$$

Therefore,

$$u(x) = \int_0^x \frac{1}{c} y f(y)\,dy + \int_x^1 \frac{1}{c} x f(y)\,dy$$

$$\Rightarrow u'(x) = \frac{1}{c} x f(x) + \frac{1}{c}\int_x^1 f(y)\,dy - \frac{x}{c} f(x)$$

$$= \frac{1}{c}\int_x^1 f(y)\,dy$$

$$\Rightarrow u''(x) = -\frac{1}{c} f(x).$$

$\blacksquare$

6.2.11

Let $w>0$. Find the Green's function for

$$\begin{cases} -u''+w^2u=f(x) \\ u(0)=0 \\ u'(1)=0 \end{cases}$$

Use your Green's function to find the solution when

$$f(x)=\begin{cases} 1, & 0\leq x\leq 1/2 \\ -1, & 1/2\leq x\leq 1 \end{cases}$$

Solution:

The Green's function satisfies

$$-G''(x,y)+w^2G(x,y)=\delta(x-y)$$

Boundary Conditions imply:

$$G(x,y)=\begin{cases} A\sinh(wx), & x<y \\ B\cosh(w(1-x)), & x>y \end{cases}$$

Continuity implies:

$$A=\frac{B\cosh(w(1-y))}{\sinh(wy)}$$

The jump condition implies:

$$-wB\sinh(w(1-y))-Aw\cosh(wy)=-1$$

$$\Rightarrow -wB(\sinh(w(1-y))+\cosh(w(1-y))\coth(wy))=-1$$

$$\Rightarrow B=\frac{\sinh(wy)}{(\sinh(wy)\sinh(w(1-y))+\cosh(w(1-y))\cosh(wy))w}$$

$$=\frac{\sinh(wy)}{\cosh(w)w}$$

$$\Rightarrow G(x,y)=\begin{cases} \dfrac{\cosh(w(1-y))}{\cosh(w)w}\sinh(wx), & x<y \\ \dfrac{\sinh(wy)}{\cosh(w)w}\cosh(w(1-x)), & x>y. \end{cases}$$

#6.2.12

Suppose $w>0$. Does the Neumann boundary value problem

$$-v''+w^2v=f(x)$$
$$v'(0)=v'(1)=0.$$

admit a Green's function?

Solution:

The green's function would satisfy:

$$-G''(x,y)+w^2G(x,y)=\delta(x-y)$$

Boundary conditions imply

$$G(x,y)=\begin{cases} A\cosh(wx), & x<y \\ B\cosh(w(1-x)), & x>y \end{cases}$$

Continuity implies:

$$A\cosh(wy)=B\cosh(w(1-y))$$

The jump condition implies:

$$wB\sinh(w(1-y))+wA\sinh(wy)=1$$

$$\Rightarrow wB\left(\sinh(w(1-y))+\frac{\cosh(w(1-y))\sinh(wy)}{\cosh(wy)}\right)=1$$

$$\Rightarrow wB(\cosh(wy)\sinh(w(1-y))+\cosh(w(1-y))\sinh(wy))=\cosh(wy)$$

$$\Rightarrow B=\frac{\cosh(wy)}{w\sinh(w)}, \quad A=\frac{\cosh(w(1-y))}{w\sinh(w)}$$

$$\Rightarrow G(x,y)=\frac{1}{w\sinh(w)}\begin{cases} \cosh(w(1-y))\cosh(wx), & x<y \\ \cosh(wy)\cosh(w(1-x)) & x>y \end{cases}$$

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Therefore,

$$u(x) = \frac{\cosh(w(1-x))}{\cosh(w)w}\int_0^x \sinh(wy)f(y)\,dy + \frac{\sinh(wx)}{\cosh(w)w}\int_x^1 \cosh(w(1-y))f(y)\,dy$$

Consequently, if $x<1/2$ it follows that

$$u(x)=\frac{\cosh(w(1-x))}{\cosh(w)w}\int_0^x \sinh(wy)\,dy+\frac{\sinh(wx)}{\cosh(w)w}\int_x^{1/2}\cosh(w(1-y))\,dy$$
$$-\frac{\sinh(wx)}{\cosh(w)w}\int_x^1 \cosh(w(1-y))\,dy$$

$$=\frac{\cosh(w(1-x))}{\cosh(w)w^2}\left(\cosh(wx)-1\right)-\frac{\sinh(wx)}{\cosh(w)w^2}\left(\sinh\left(\frac{w}{2}\right)-\sinh(w(1-x))\right)$$
$$+\frac{\sinh(wx)}{\cosh(w)w^2}\sinh(w(1-x))$$

$$=\frac{1}{\cosh(w)w^2}\left(\cosh(wx)-1-\sinh(wx)\sinh(w/2)\right)$$

If $x>1/2$ it follows that

$$u(x)=\frac{\cosh(w(1-x))}{\cosh(w)w}\int_0^{1/2}\sinh(wy)\,dy-\frac{\cosh(w(1-x))}{\cosh(w)w}\int_{1/2}^x \sinh(wy)\,dy$$
$$-\frac{\sinh(wx)}{\cosh(w)w}\int_x^1 \cosh(w(1-y))\,dy$$

$$=\frac{\cosh(w(1-x))}{\cosh(w)w^2}\left(\cosh\left(\frac{w}{2}\right)-1\right)-\frac{\cosh(w(1-x))}{\cosh(w)w^2}\left(\cosh(wx)-\cosh\left(\frac{w}{2}\right)\right)$$
$$+\frac{\sinh(wx)}{\cosh(w)w^2}\sinh(w(1-x))$$

$\blacksquare$

#6.2.10

Solve the boundary value problem

$$-u'' + w^2 u = \begin{cases} 1, & 0<x<1/2 \\ -1, & 1/2<x<1 \end{cases}$$

$$u(0)=u(1)=0.$$

Solution:

If $x<1/2$ then

$$u(x) = \frac{1}{w\sinh(w)}\Big(\int_0^x \sinh(wx)\sinh(w(1-y))\,dy + \int_x^{1/2} \sinh(w(1-x))\sinh(wy)\,dy$$
$$- \int_{1/2}^1 \sinh(w(1-x))\sinh(wy)\,dy\Big)$$

$$= \frac{1}{w^2\sinh(w)}\Big(\sinh(wx)(\cosh(w) - \cosh(w(1-x)) + \cosh(w/2)\sinh(w(1-x))$$
$$- \sinh(w(1-x))\cosh(wx)\Big) - \sinh(w(1-x)\cosh(w) + \sinh(w(1-x))\text{co}$$

If $x>1/2$ then

$$u(x) = \frac{1}{w\sinh(w)}\Big(\int_0^{1/2} \sinh(wx)\sinh(w(1-y))\,dy - \int_{1/2}^x \sinh(wx)\sinh(w(1-y))\,dy$$
$$- \int_x^1 \sinh(w(1-x))\sinh(wy)\,dy\Big)$$

$$= \frac{1}{w^2\sinh(w)}\Big(-\sinh(wx)(\cosh(\tfrac{w}{2}) - \cosh(w)) + \sinh(wx)(\cosh(w(1-x) - \cosh(\tfrac{w}{2}))$$
$$- \sinh(w(1-x))(\sinh(w) - \sinh(x)\Big)$$

:6.2.13.

Prove the addition formulas for sinh(x) and cosh(x).

Solution:

1. $\sinh(x)\cosh(y) = \left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)$

$$= \frac{e^{x+y}-e^{y-x}+e^{x-y}-e^{-x-y}}{4}$$

Therefore,

$$\sinh(\alpha)\cosh(\beta)+\sinh(\beta)\cosh(\alpha) = \frac{e^{\alpha+\beta}-e^{-\alpha-\beta}}{2} = \sinh(\alpha+\beta).$$

2. $\cosh(x)\cosh(y) = \left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)$

$$= \frac{e^{x+y}+e^{y-x}+e^{x-y}+e^{-x-y}}{4}$$

$$\sinh(x)\sinh(y) = \left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$

$$= \frac{e^{x+y}-e^{y-x}-e^{x-y}+e^{-x-y}}{4}$$

$$\Rightarrow \cosh(\alpha)\cosh(\beta)+\sinh(\alpha)\sinh(\beta) = \frac{e^{\alpha+\beta}+e^{-\alpha-\beta}}{2} = \cosh(\alpha+\beta).$$

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:6.2.14

Prove that

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} F(x,y)\,dy = F(x,\beta(x))\frac{d\beta}{dx} - F(x,\alpha(x))\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial F}{\partial x}\,dy$$

~~proof~~:

Let $H(x,y)$ satisfy $\frac{\partial H}{\partial y} = F$. Therefore,

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} F(x,y)\,dy = \frac{d}{dx}\left(H(x,\beta(x)) - H(x,\alpha(x))\right)$$

$$= \left.\frac{\partial H}{\partial x}\right|_{y=\beta(x)} + \frac{\partial H}{\partial y}\frac{d\beta}{dx} - \left.\frac{\partial H}{\partial x}\right|_{y=\alpha(x)} - \frac{\partial H}{\partial y}\frac{d\alpha}{dx}$$

$$= F(x,\beta(x))\frac{d\beta}{dx} - F(x,\alpha(x))\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} F(x,y)\,dy.$$

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Graduate Problem

Find the Green's function for the following boundary value problem:

$$u''''(x) = f(x)$$

$$u(0) = u(1) = 0$$

$$u'(0) = u'(1) = 0$$

Solution:

The Green's function satisfies:

$$G''''(x,y) = \delta(x-y).$$

Boundary conditions imply that

$$G(x,y) = \begin{cases} Ax^2 + Bx^3, & x<y \\ C(1-x)^2 + D(1-x)^3, & x>y. \end{cases}$$

The jump condition in the third derivative implies:

$$-6D - 6B = 1$$

$$\Rightarrow B = -D - \frac{1}{6}.$$

Continuity of second derivatives implies

$$2A + 6By = 2C + 6D(1-y)$$

$$\Rightarrow 2A + 6\left(-D - \frac{1}{6}\right)y = 2C + 6D - 6Dy$$

$$\Rightarrow 2A - y = 2C + 6D$$

$$\Rightarrow A = C + 3d + \frac{y}{2}$$

Continuity of first derivatives implies

$$2Ay + 3By^2 = -2C(1-y) - 3D(1-y)^2$$

$$\Rightarrow C = -\frac{3}{2}D - \frac{y^2}{4}.$$

Continuity implies:

$$Ay^2 + By^3 = C(1-y)^2 + D(1-y)^3$$

$$\Rightarrow A = \frac{1}{2}y(1-y)^2, \quad C = \frac{1}{2}y^2(1-y)$$

$$B = -\frac{1}{6}(1+2y)(1-y)^2, \quad D = \frac{1}{6}(2y-3)y^2$$

Therefore,

$$G(x,y) = \begin{cases} \frac{1}{2} y(1-y)^2 x^2 - \frac{1}{6}(1+2y)(1-y)^2 x^3, & x<y \\ \frac{1}{2} y^2(1-y)(1-x)^2 + \frac{y^2}{6}(2y-3)(1-x)^3, & x>y \end{cases}$$

$$= \begin{cases} \frac{x^2(1-y)^2}{2}\left(y - \frac{1}{3}(1+2y)x \right), & x<y \\ \frac{y^2(1-x)^2}{2}\left(1-y + \frac{1}{3}(2y-3)(1-x) \right), & x>y. \end{cases}$$

$$= \begin{cases} \frac{x^2(1-y)^2}{6}\left(3y - x - 2yx \right), & x<y \\ \frac{y^2(1-x)^2}{6}\left(3x - y - 2yx \right), & x>y. \end{cases}$$

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