MST 352/652
Homework #12

Due Date: April 23, 2019

1 Problems for everyone

1. pg. 273-274, #7.1.1-7.1.4, #7.1.12, #7.1.15
2. Compute $\mathcal{F}[xe^{-ax^2}](k)$, where $a > 0$ is a constant.
3. Given that $\mathcal{F}[xe^{-|x|}](k) = -\dfrac{4ik}{(1+k^2)^2}$, find $\mathcal{F}\left[\dfrac{x}{(1+x^2)^2}\right](k)$.
4. pg. 277-278, #7.2.1, #7.2.8-7.2.9.
5. pg. 280-281, #7.3.4-7.3.5, #7.3.7.
6. pg. 283-384, #7.3.10-7.3.12

2 Graduate Problems

1. pg. 273-274, #7.1.8, #7.1.19
2. pg. 283-384, #7.3.13, #7.3.20

Homework #12

#7.1.1

Find the Fourier transforms of the following functions

a.) $e^{-(x+4)^2}$

c.) $\begin{cases} x, & |x|<1 \\ 0, & \text{o.w.} \end{cases}$

e.) $\begin{cases} e^{-|x|}, & |x|>1 \\ e^{-1}, & |x|\le 1 \end{cases}$

g.) $\begin{cases} 1-|x|, & |x|\le 1 \\ 0, & \text{o.w.} \end{cases}$

Solution:

a.) $$\mathcal{F}[e^{-(x+4)^2}](k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+4)}e^{-ikx}dx$$

$$=\frac{1}{\sqrt{2\pi}}e^{4ik}\int_{-\infty}^{\infty} e^{-u^2}e^{-iku}du$$

$$=\frac{1}{\sqrt{2}}e^{4ik}e^{-k^2/4}.$$

c.) $$\mathcal{F}[f(x)](k)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} xe^{-ikx}dx$$

$$=\frac{i}{\sqrt{2\pi}}2\int_0^1 -x\sin(kx)dx$$

$$=\frac{2}{\sqrt{2\pi}}\left(\frac{x\cos(kx)}{k}\Big|_0^1-\int_0^1\frac{1}{k}\cos(kx)dx\right)$$

$$=\frac{2i}{\sqrt{2\pi}}\left(\frac{\cos(k)}{k}-\frac{\sin(k)}{k^2}\right)$$

e.) $$\mathcal{F}[f(x)](k)=\frac{1}{\sqrt{2\pi}}\left(\int_{-1}^{1}e^{-1}e^{-ikx}dx+2\int_1^{\infty}e^{-x}\cos(kx)dx\right)$$

$$=\frac{2\sin(k)}{\sqrt{2\pi}ek}+\frac{2}{\sqrt{2\pi}}\left(\frac{\cos(k)-k\sin(k)}{ek^2+e}\right.$$

g.) $\mathcal{F}[f(x)](k) = \frac{2}{\sqrt{2\pi}}\int_0^1 (1-x)\cos(kx)\,dx$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1-\cos(k)}{k^2}.$$

#2.

Compute $\mathcal{F}[xe^{-ax^2}]$, where $a>0$ is a constant.

Solution:

$$\mathcal{F}[xe^{-ax^2}](k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-ax^2}e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1}{-i}e^{-ax^2}\frac{d}{dk}e^{-ikx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\, i\frac{d}{dk}\int_{-\infty}^{\infty} e^{-ax^2}e^{-ikx}dx$$

$$= i\frac{1}{\sqrt{2a}}\frac{d}{dk}e^{-k^2/4a}$$

$$= \frac{i}{\sqrt{2a}}\,\frac{-2k}{4a}e^{-k^2/4a}$$

$$= -\frac{i\sqrt{2}}{4a^{3/2}}ke^{-k^2/4a}$$

#3.

Given that $\mathcal{F}[xe^{-|x|}](k) = \frac{1}{\sqrt{2\pi}}\left(\frac{-4ik}{(1+k^2)^2}\right)$, find $\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right](k)$.

Solution:

$$\mathcal{F}\left[\frac{x}{(1+x^2)^2}\right](k) = \mathcal{F}\left[\frac{x}{(1+x^2)^2}\right](-k)$$

$$= \frac{\sqrt{2\pi}}{-4i}(-k)e^{-|k|}$$

$$= \frac{\sqrt{2\pi}}{4}ike^{-|k|}.$$

#7.3.5.

Use the Fourier transform to solve

$$-u'' + u = \delta'(x-1).$$

Solution:

Taking the Fourier transform it follows that:

$$(1+k^2)\hat{u} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta'(x-1)e^{-ikx}dx$$

$$= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-1)(-ik)e^{-ikx}dx$$

$$= \frac{ik}{\sqrt{2\pi}}e^{-ik}$$

$$\Rightarrow \hat{u} = \frac{ik}{\sqrt{2\pi}}\frac{e^{-ik}}{1+k^2}$$

$$\Rightarrow u = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{k}{1+k^2}e^{-ik}e^{ikx}dk$$

$$= \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+k^2}e^{-ik}\frac{1}{i}\frac{d}{dx}e^{ikx}dk$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{1}{1+k^2}e^{ik(x-1)}dk$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}e^{-|x-1|}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dx}\begin{cases} e^{-(x-1)}, & x>1 \\ e^{-(1-x)}, & x\le 1\end{cases}$$

$$= \frac{1}{\sqrt{2\pi}}\begin{cases} -e^{-(x-1)}, & x>1 \\ e^{-(1-x)}, & x\le 1\end{cases}$$

$$= \frac{1}{\sqrt{2\pi}}\operatorname{sgn}(1-x)e^{-|x-1|}.$$

#7.3.4

Find a solution to the differential equation $-u''+4u=\delta(x)$ by using the Fourier transform.

Solution:

Taking the Fourier transform it follows that

$$(k^2+4)\hat{u}=\frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \hat{u}=\frac{1}{\sqrt{2\pi}}\frac{1}{4+k^2}$$

$$=\frac{1}{4}\sqrt{\frac{2}{\pi}}\cdot\frac{2}{4+k^2}$$

$$\Rightarrow \hat{u}=\frac{1}{4}e^{-2|x|}.$$

#7.3.11

What is the convolution of e^{-x^2} with itself?

Solution:

Let $h(x)=e^{-x^2}*e^{-x^2}$. Therefore,

$$\hat{h}=\sqrt{2\pi}\cdot\frac{1}{\sqrt{2}}e^{-k^2/4}\cdot\frac{1}{\sqrt{2}}e^{-k^2/4}$$

$$=\sqrt{\frac{\pi}{2}}e^{-k^2/2}$$

$$\rightarrow h(x)=\sqrt{\frac{\pi}{2}}\cdot\sqrt{2}\,e^{-x^2/4/2}$$

$$=\sqrt{\pi}\,e^{-x^2/2}.$$

#7.3.13.

a.) Write down the Fourier transform of the box function

$$f(x)=\begin{cases}1, & |x|<1/2\\ 0, & \text{o.w.}\end{cases}$$

b.) Graph the hat function $h(x)=(f*f)(x)$ and find its Fourier transform.

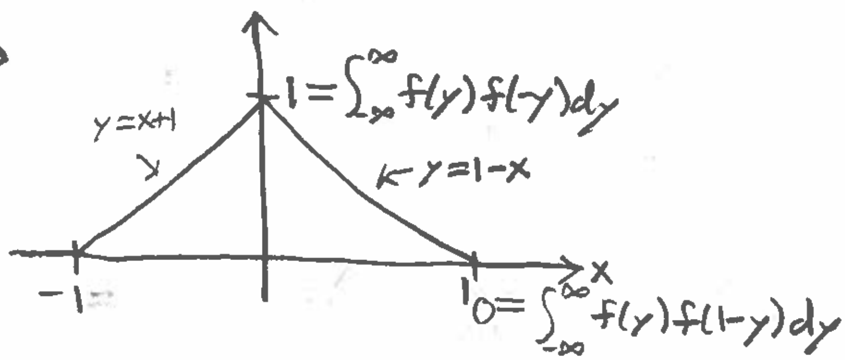
c.) Determine the cubic B spline $s(x)=h*h(x)$ and its Fourier transform.

__Solution:__

a.)
$$\hat{f}(k)=\frac{2}{\sqrt{2\pi}}\int_0^{1/2}\cos(kx)\,dx$$
$$=\frac{2}{\sqrt{2\pi}\,k}\sin(k/2)$$
$$=\sqrt{\frac{2}{\pi}}\,\frac{\sin\left(\frac{k}{2}\right)}{k}$$

b.)
$$h(x)=(f*f)(x)$$
$$=\int_{-\infty}^{\infty} f(y)f(x-y)\,dy$$

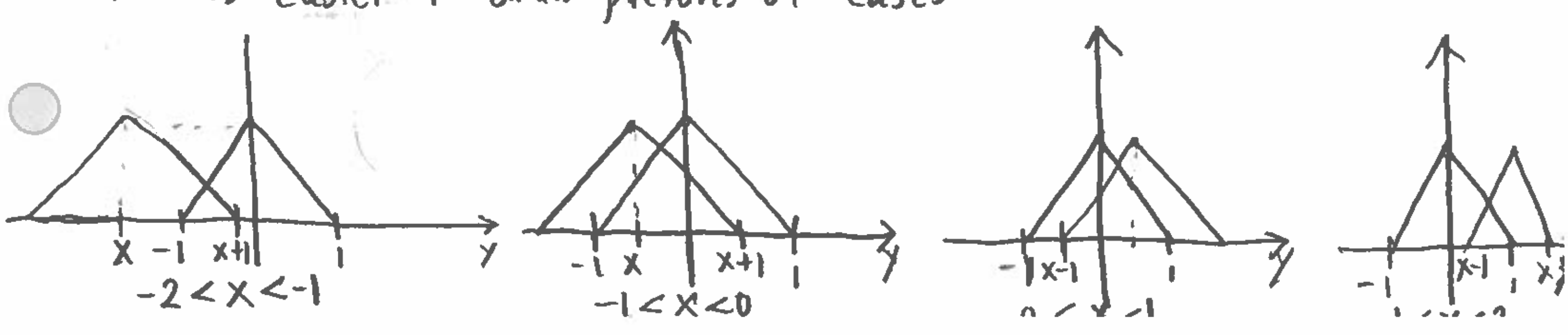
$\Rightarrow$



$$\hat{h}=\sqrt{2\pi}\,\hat{f}\cdot\hat{f}$$
$$=\sqrt{2\pi}\,\frac{\sin^2(k/2)}{k^2}\cdot\frac{2}{\pi}$$
$$=\frac{4}{\sqrt{2\pi}}\,\frac{\sin^2(k/2)}{k^2}$$

c.)
$$(h*h)(x)=\int_{-\infty}^{\infty} h(x-y)h(y)\,dy$$

It is easier to draw pictures of cases



Graduate Problems.

#7.1.8

a.) Find the Fourier transform of the hat function.

$$f_n(x)=\begin{cases} n-n^2|x|, & |x|\le 1/n \\ 0, & \text{o.w.} \end{cases}$$

b.) What is the limit as $n\to\infty$, of $\hat{f}_n(k)$?

c.) In what sense is the limit the Fourier transform of the limit of $f_n(x)$?

Solution:

a.) $$\mathcal{F}[f_n(x)]=\frac{2}{\sqrt{2\pi}}\int_0^{1/n}(n-n^2x)\cos(kx)\,dx$$

$$=\frac{2}{\sqrt{2\pi}}\left(\frac{(n-n^2x)\sin(kx)}{k}\Big|_0^{1/n}+\frac{1}{k}\int_0^{1/n}n^2\sin(kx)dx\right)$$

$$=\frac{2}{\sqrt{2\pi}}\left(-\frac{n^2}{k^2}\cos(kx)\Big|_0^{1/n}\right)$$

$$=\frac{2}{\sqrt{2\pi}}\frac{n^2}{k^2}\left(1-\cos\left(\frac{k}{n}\right)\right)$$

Note, $\lim\limits_{k\to 0}\frac{2}{\sqrt{2\pi}}\frac{n^2}{k^2}\left(\frac{k^2}{2n^2}-\frac{k^4}{24n^4}+\dots\right)=\frac{1}{\sqrt{2\pi}}$.

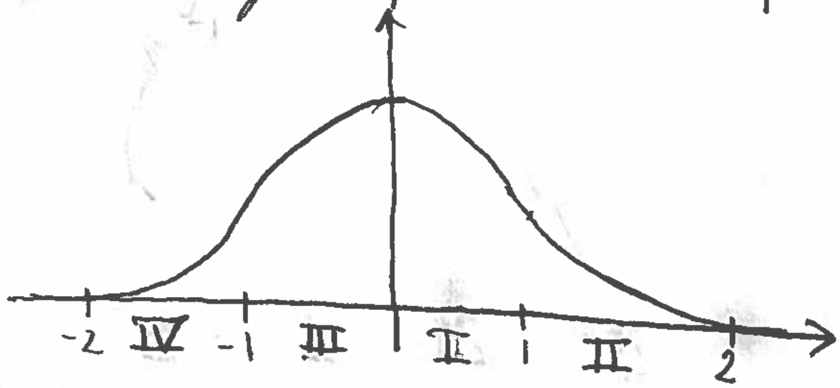
b.) $$\lim_{n\to\infty}\frac{2}{\sqrt{2\pi}}\hat{f}_n(k)=\lim_{n\to\infty}\frac{2}{\sqrt{2\pi}}\frac{n^2}{k^2}\left(\frac{k^2}{2n^2}-\frac{k^4}{24n^4}+\dots\right)=\frac{1}{\sqrt{2\pi}}.$$

c.) $f_n(x)$ satisfies the following:

1. If $x\neq 0$ then $\lim\limits_{n\to\infty}f_n(x)=0$.
2. If $x=0$ then $\lim\limits_{n\to\infty}f_n(x)=\infty$
3. $\int_{-\infty}^{\infty}f_n(x)=\frac{1}{2}\cdot\frac{2}{n}\cdot n=1.$

Therefore, in a distributional sense $f_n(x)\to\delta(x)$. Moreover, $\hat{\delta}(x)=\frac{1}{\sqrt{2\pi}}$.

The resulting shape is a cubic pulse:



Let

$$S(x) = \begin{cases} S_1(x), & 0<x<1 \\ S_2(x), & 1<x<2 \\ S_1(-x), & -1<x<0 \\ S_2(-x), & -2<x<1 \end{cases},$$

where

$$S_1(x) = ax^3+bx^2+cx+d.$$
$$S_2(x) = e(x-2)^2(x-f)$$

We also know,

1. $S_1(0) = \int_{-1}^{1} h(y)h(-y)dy$

$$= 2\int_0^1 (1-x)^2 dx$$

$$= \left. \frac{-2(1-x)^3}{3} \right|_0^1$$

$$= \frac{2}{3}$$

$\Rightarrow d = 2/3.$

2. $S_1'(0) = 0$

$\Rightarrow c = 0.$

3. $S_1''(1) = 0$

$\Rightarrow 6a+2b = 0$

$\Rightarrow b = -3a$

4. $S_1(1) = \int_0^1 (1-x)x\,dx = \frac{1}{6}.$

$\Rightarrow a - 3a + \frac{2}{3} = \frac{1}{6} \Rightarrow a = \frac{1}{4} \Rightarrow b = -\frac{3}{4}.$

Therefore,

$$S_1(x) = \frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{2}{3}.$$

To calculate $S_2(x)$ we have that

$$S_2''(1) = 0 \quad \text{and} \quad S_2'(1) = \frac{1}{6}$$

$$\Longrightarrow S_2(x) = \frac{1}{12}(x-2)^2(x+1).$$

#7.3.20

Suppose $f(x)$ and $g(x)$ are 0 for all $x<0$. Prove that

$$h = (f*g)(x) = \begin{cases} \int_0^x f(x-y)g(y), & x>0 \\ 0, & x\leq 0. \end{cases}$$

Solution:

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{0}^{\infty} f(x-y)g(y)\,dy = \int_{\substack{x-y>0 \\ y>0}} f(x-y)g(y)\,dy$$

$$\Longrightarrow h(x) = \int_{\substack{x>y \\ y>0}} f(x-y)g(y)\,dy$$

$$= \int_0^x f(x-y)g(y)\,dy.$$