MST 352/652
Homework #13

Due Date: April 30, 2019

1 Problems for everyone

1. Consider the following initial value problem for the heat equation with proportional heat loss:

$$u_t = Du_{xx} - au, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = e^{-x^2},$$

where $D > 0$ and $a > 0$ are constants.

 (a) Using Fourier transforms find a formula for the solution to this initial value problem.
 (b) Explain what effect the additional term $-au$ has on the behavior of solutions compared with the heat equation without loss.
 (c) Show that by changing variables $v(x, t) = e^{at}u(x, t)$ that v satisfies the heat equation $v_t = Dv_{xx}$.

2. Consider the following initial value problem for the heat equation with advection:

$$u_t = Du_{xx} - cu_x, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = e^{-x^2},$$

where $D > 0$ and $c > 0$ are constants.

 (a) Using Fourier transforms find a formula for the solution to this initial value problem.
 (b) Assuming $D = 1$ and $c = 1$, on the same axis sketch $u(x, 0)$, $u(x, 1)$, and $u(x, 2)$ as functions of x. Explain qualitatively the behavior of the solution as time increases.
 (c) By changing variables to $\tau = t$ and $X = x - ct$ show that u satisfies $u_\tau = Du_{XX}$.

3. Consider the following initial value problem for the heat equation:

$$u_t = Du_{xx}, \ x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = f(x),$$

where $D > 0$ is a constant. Show that if $f(x)$ is an odd function then $u(x, t)$ is an odd function in the variable x.

4. Consider the following initial boundary value problem for the heat equation on the half line:

$$u_t = Du_{xx}, \ x \in [0, \infty), \ t > 0,$$
$$u(x, 0) = f(x) \text{ and } u(0, t) = 0,$$

where $D > 0$ is a constant.

(a) What do the boundary conditions model in this situation?

(b) Solve this initial value problem by extending $f(x)$ to an *odd* function on the entire real axis.

5. Consider the following initial boundary value problem for the heat equation on the half line:

$$\begin{aligned} &u_t = Du_{xx},\ x \in [0, \infty),\ t > 0, \\ &u_x(0,t) = 0, \\ &u(x,0) = f(x), \end{aligned}$$

where $D > 0$ is a constant. Solve this initial value problem by extending $f(x)$ to an *even* function on the entire real axis.

6. Using Duhamel's principle or Fourier transforms, find a formula for the solution to the initial value problem for the convection equation:

$$\begin{aligned} &u_t + u_x = f(x,t),\ x \in \mathbb{R},\ t > 0 \\ &u(x,0) = 0. \end{aligned}$$

7. Solve the problem

$$\begin{aligned} &u_t + 2u_x = xe^{-t},\ x \in \mathbb{R},\ t > 0 \\ &u(x,0) = 0 \end{aligned}$$

8. Solve the following initial value problem for the heat equation with a source:

$$\begin{aligned} &u_t = Du_{xx} + (1 - 2x^2)e^{-x^2},\ x \in \mathbb{R},\ t > 0, \\ &u(x,0) = e^{-ax^2}, \end{aligned}$$

where $D, a > 0$.

2 Graduate Problems

1. Consider the following initial value problem for Schrödinger's equation:

$$\begin{aligned} &\psi_t = i\psi_{xx} - iV(x)\psi,\ x \in \mathbb{R},\ t > 0, \\ &\psi(x,0) = f(x), \end{aligned}$$

where V is a real valued function satisfying $\lim_{|x|\to\infty} V(x) = 0$.

(a) Show that if ψ solves this initial value problem then the quantity $P(t)$ defined by

$$P(t) = \int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,dx,$$

where * denotes complex conjugation, is constant in time.

(b) Solve this initial value problem assuming $V = 0$ and $f(x) = (2/\pi)^{1/4}e^{-x^2}$.

(c) In quantum mechanics the quantity $p(x,t) = \psi^*(x,t)\psi(x,t)$ models the probability distribution for the location of a particle. For your answer to part (b), compute $p(x,t)$ and plot this function at times $t = 0$, $t = 1$ and $t = 2$. What does this result imply about the location of the particle?