MST 352/652
Homework #13

Due Date: April 30, 2019

1 Problems for everyone

1. Consider the following initial value problem for the heat equation with proportional heat loss:

$$u_t = Du_{xx} - au, \; x \in \mathbb{R}, \; t > 0,$$
$$u(x,0) = e^{-x^2},$$

where $D > 0$ and $a > 0$ are constants.

   (a) Using Fourier transforms find a formula for the solution to this initial value problem.

   (b) Explain what effect the additional term $-au$ has on the behavior of solutions compared with the heat equation without loss.

   (c) Show that by changing variables $v(x,t) = e^{at}u(x,t)$ that v satisfies the heat equation $v_t = Dv_{xx}$.

/ 2. Consider the following initial value problem for the heat equation with advection:

$$u_t = Du_{xx} - cu_x, \; x \in \mathbb{R}, \; t > 0,$$
$$u(x,0) = e^{-x^2},$$

where $D > 0$ and $c > 0$ are constants.

   (a) Using Fourier transforms find a formula for the solution to this initial value problem.

   (b) Assuming $D = 1$ and $c = 1$, on the same axis sketch $u(x,0)$, $u(x,1)$, and $u(x,2)$ as functions of x. Explain qualitatively the behavior of the solution as time increases.

   (c) By changing variables to $\tau = t$ and $X = x - ct$ show that u satisfies $u_\tau = Du_{XX}$.

/ 3. Consider the following initial value problem for the heat equation:

$$u_t = Du_{xx}, \; x \in \mathbb{R}, \; t > 0,$$
$$u(x,0) = f(x),$$

where $D > 0$ is a constant. Show that if $f(x)$ is an odd function then $u(x,t)$ is an odd function in the variable x.

4. Consider the following initial boundary value problem for the heat equation on the half line:

$$u_t = Du_{xx}, \; x \in [0,\infty), \; t > 0,$$
$$u(x,0) = f(x) \text{ and } u(0,t) = 0,$$

where $D > 0$ is a constant.

(a) What do the boundary conditions model in this situation?

(b) Solve this initial value problem by extending $f(x)$ to an *odd* function on the entire real axis.

5. Consider the following initial boundary value problem for the heat equation on the half line:

$$\begin{aligned} &u_t = Du_{xx}, \; x \in [0,\infty), \; t > 0, \\ &u_x(0,t) = 0, \\ &u(x,0) = f(x), \end{aligned}$$

where $D > 0$ is a constant. Solve this initial value problem by extending $f(x)$ to an *even* function on the entire real axis.

6. Using Duhamel's principle or Fourier transforms, find a formula for the solution to the initial value problem for the convection equation:

$$\begin{aligned} &u_t + u_x = f(x,t), \; x \in \mathbb{R}, \; t > 0 \\ &u(x,0) = 0. \end{aligned}$$

7. Solve the problem

$$\begin{aligned} &u_t + 2u_x = xe^{-t}, \; x \in \mathbb{R}, \; t > 0 \\ &u(x,0) = 0 \end{aligned}$$

8. Solve the following initial value problem for the heat equation with a source:

$$\begin{aligned} &u_t = Du_{xx} + (1-2x^2)e^{-x^2}, \; x \in \mathbb{R}, \; t > 0, \\ &u(x,0) = e^{-ax^2}, \end{aligned}$$

where $D, a > 0$.

2 Graduate Problems

1. Consider the following initial value problem for Schrödinger's equation:

$$\begin{aligned} &\psi_t = i\psi_{xx} - iV(x)\psi, \; x \in \mathbb{R}, \; t > 0, \\ &\psi(x,0) = f(x), \end{aligned}$$

where V is a real valued function satisfying $\lim_{|x|\to\infty} V(x) = 0$.

(a) Show that if ψ solves this initial value problem then the quantity $P(t)$ defined by

$$P(t) = \int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,dx,$$

where * denotes complex conjugation, is constant in time.

(b) Solve this initial value problem assuming $V = 0$ and $f(x) = (2/\pi)^{1/4}e^{-x^2}$.

(c) In quantum mechanics the quantity $p(x,t) = \psi^*(x,t)\psi(x,t)$ models the probability distribution for the location of a particle. For your answer to part (b), compute $p(x,t)$ and plot this function at times $t = 0$, $t = 1$ and $t = 2$. What does this result imply about the location of the particle?

Homework #13

#2

Consider the following initial value problem for the heat equation with advection:

$$U_t = D U_{xx} - c U_x, \quad x \in \mathbb{R},\ t>0,$$
$$U(x,0) = e^{-x^2},$$

where $D>0$ and $c>0$ are constants.

a.) Using Fourier transforms find a formula for this solution to this initial value problem.

b.) Assuming $D=1$ and $c=1$, on the same axis sketch $U(x,0)$, $U(x,1)$, and $U(x,2)$. Explain qualitatively the behavior of the solution as time increases.

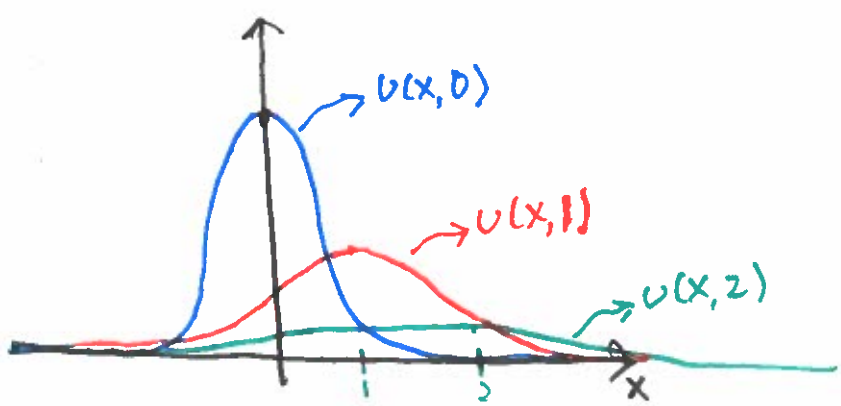
C.) By changing variables to $\tau = t$ and $X = x - ct$ show that $U_\tau = D U_{XX}$.

Solution:

a.) Taking the Fourier transform it follows that

$$\hat{U}_t = -(k^2 + ikc)\hat{U}$$
$$\hat{U}(k,0) = \mathcal{F}[e^{-x^2}](k) = \hat{U}_0$$
$$\Rightarrow \hat{U} = \hat{U}_0 \exp(-(k^2+ikc)t)$$
$$\Rightarrow U(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(x,0) * \mathcal{F}^{-1}\left[\exp(-(k^2+ikc)t)\right]$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-y^2} \cdot e^{-(x-ct-y)^2/4t} dy$$
$$= \frac{1}{\sqrt{1+4t}} \exp\left(\frac{-(x-ct)^2}{1+4t}\right).$$

b.)



C.) The initial heat is diffusing and being advected to the right at speed 1.

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#3.

Consider the following initial value problem for the heat equation:

$$U_t = D U_{xx}$$
$$u(x,0) = f(x),$$

where $D>0$ is a constant. Show that if $f(x)$ is an odd function then $u(x,t)$ is an odd function in x.

Solution:

The solution to this initial value problem is given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy.$$

Therefore,

$$u(-x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(y) \exp\left(\frac{-(-x-y)^2}{4Dt}\right) dy$$
$$= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x+y)^2}{4Dt}\right) dy$$
$$= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(-y) \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy$$
$$= -\frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(x-y)^2}{4Dt}\right) dy$$
$$= -u(x,t).$$

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#5.

Consider the following initial value problem for the heat equation on the half line:

$$u_t = D u_{xx}$$
$$u_x(0,t) = 0$$
$$u(x,0) = f(x),$$

where $D>0$ is a constant. Solve this initial value problem by extending $f(x)$ to an even function on $\mathbb{R}$.

Solution:

Let $\tilde{f}(x)$ be defined by:

$$\tilde{f}(x) = \begin{cases} f(-x), & x<0 \\ f(x), & x \geq 0 \end{cases}$$

The solution to the initial value problem

$$\tilde{u}_t = D\tilde{u}_{xx}$$
$$\tilde{u}(x,0) = \tilde{f}(x)$$

is given by:

$$\tilde{u}(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \tilde{f}(y) e^{-(x-y)^2/4Dt} \, dy.$$

$$= \frac{1}{\sqrt{4\pi Dt}} \left(\int_{-\infty}^{0} f(-y) e^{-(x-y)^2/4Dt} \, dy + \int_{0}^{\infty} f(y) e^{-(x-y)^2/4Dt} \, dy \right)$$

$$= \frac{1}{\sqrt{4\pi Dt}} \int_{0}^{\infty} f(y) \left(e^{-(x+y)^2/4Dt} + e^{-(x-y)^2/4Dt} \right) dy.$$

Clearly, $\tilde{u}$ is even and hence $\tilde{u}_x(0,t) = 0$. Therefore, $u(x,t) = \tilde{u}(x,t)$, $x>0$ solves the PDE of interest.

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#7

Solve the problem

$$U_t + 2U_x = xe^{-t}$$
$$U(x,0) = 0.$$

Solution:

The auxiliary problem is given by:

$$W_t + 2U_x = 0$$
$$W(0,\tau,x) = xe^{-\tau}$$
$$\Rightarrow W(t,\tau,x) = (x-2t)e^{-\tau}$$

Therefore,

$$U(t,x) = \int_0^t W(t-\tau,\tau,x)\,d\tau$$
$$= \int_0^t (x-2(t-\tau))e^{-\tau}\,d\tau$$
$$= 2-2t+x-e^{-t}(2+x).$$

#8.

Solve the following intial value problem:

$$U_t = DU_{xx} + (1-2x^2)e^{-x^2}$$
$$U(x,0) = e^{-ax^2},$$

where $D, a > 0$.

Solution:

Consider the two problems

$$V_t = DV_{xx} \qquad , \quad W_t = DW_{xx} + (1-2x^2)e^{-x^2}$$
$$V(x,0) = e^{-ax^2} \qquad \quad W(x,0) = 0$$

We can solve for V directly:

$$\hat{V}_t = -K^2 D\hat{V}$$
$$\hat{V}(K,0) = \frac{1}{\sqrt{2a}}e^{-K^2/4a}$$
$$\Rightarrow \hat{V}(K,t) = \frac{1}{\sqrt{2a}}e^{-K^2/4a}e^{-K^2Dt}$$
$$\Rightarrow V(x,t) = \frac{1}{\sqrt{a^2+4aDt}}e^{-x^2/a+4Dt}$$

W is found using Duhamel's principle with the following auxilary problem:

$$\varphi_t = D\varphi_{xx}$$
$$\varphi(x,0,\tau) = (1-2x^2)e^{-x^2}$$
$$\Rightarrow \varphi(x,t,\tau) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}(1-2y^2)e^{-y^2}e^{-(x-y)^2/4Dt}\,dy.$$

Therefore,

$$U(x,t) = V(x,t) + W(x,t)$$
$$= \frac{1}{\sqrt{(a^2+4aDt)\pi}}e^{-x^2/a+4Dt} + \int_0^t\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi(t-\tau)}}(1-2y^2)e^{-y^2}e^{-(x-y)^2/4D(t-\tau)}\,d\tau$$

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Graduate Problems

#1.

Consider the following initial value problem

$$\psi_t = i\psi_{xx} - iV(x)\psi$$
$$\psi(x,0) = f(x)$$

Where V is a real valued function satisfying $\lim_{|x|\to\infty} V(x) = 0$.

a.) Show that P defined by

$$P(t) = \int_{-\infty}^{\infty}\psi^*(x,t)\psi(x,t)\,dx$$

is constant in time.

b.) Solve this problem assuming $V=0$ and $f(x) = \left(\frac{2}{\pi}\right)^{1/4}e^{-x^2}$.

Solution:

a.) $$\frac{dP}{dt} = \int_{-\infty}^{\infty}\left(\frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t}\right)dx$$
$$= \int_{-\infty}^{\infty}\left[\left(-i\psi^*_{xx} + iV(x)\psi^*\right)\psi + \psi^*\left(i\psi_{xx} - iV(x)\psi\right)\right]dx$$
$$= \int_{-\infty}^{\infty}\left[-i\psi^*_{xx}\psi + i\psi^*\psi_{xx}\right]dx$$
$$= \int_{-\infty}^{\infty}\left(i\psi^*_x\psi_x - i\psi^*_x\psi_x\right)dx$$
$$= 0.$$

b.) Taking Fourier transforms it follows that

$$\hat{U}_t = -ik^2\hat{U}$$

$$\hat{U}(k,0) = \left(\frac{2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2}}e^{-k^2/4}$$

$$\Rightarrow \hat{U}(k,t) = \left(\frac{2}{\pi}\right)^{1/4}\cdot\frac{1}{\sqrt{2}}e^{-k^2/4}e^{-ik^2t}$$

$$= \left(\frac{2}{\pi}\right)^{1/4}\cdot\frac{1}{\sqrt{2}}e^{-\left(\frac{1-4it}{4}\right)k^2}$$

$$\Rightarrow U(x,t) = \left(\frac{2}{\pi}\right)^{1/4}\cdot\frac{1}{\sqrt{2}}\frac{2}{\sqrt{1-4it}}e^{-x^2/1-4it}$$

c.) $$p(x,t) = U^*U = \left(\frac{2}{\pi}\right)^{1/2}\cdot\frac{4}{2\sqrt{(1-4it)(1+4it)}}e^{-x^2/(1-4it)}e^{-x^2/(1+4it)}$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{2}{\sqrt{1+16t}}e^{-x^2(1+4it+1-4it)/(1+16t)}$$

$$= \sqrt{\frac{2}{\pi}}\frac{2}{\sqrt{1+16t}}e^{-2x^2/(1+16t)}.$$