

MST 352/652

Homework #2

Due Date: January 29, 2019

1 Problems for everyone

1. pg. 18, #2.1.1-2.1.4, #2.1.6
2. pg. 23, #2.2.1-2.2.6, #2.2.8, #2.2.9
3. pg. 30, #2.2.16, #2.2.17, #2.2.20
4. Consider the problem:

$$\frac{d^2u}{dx^2} + u = 0,$$
$$u(0) = 0 \text{ and } u(L) = 0,$$

where $L > 0$ is a constant and u is a function of a single variable x . Clearly, $u(x) = 0$ is a solution. Is this solution unique, or not?

5. Solve the boundary value problem

$$u''(x) = 0,$$
$$u'(0) + ku(0) = 0 \text{ and } u'(1) \pm ku(1) = 0,$$

where $k > 0$ is a constant and u is a function of one variable. Do the + and - cases separately. What is special about the case $k = 2$?

2 Graduate Problems

Undergraduate students can complete these exercises for extra credit

1. pg. 30, #2.2.19, #2.2.25
2. Consider the equation

$$u_t + xu_x = 0$$

with the boundary condition $u(t, 0) = \phi(t)$.

- (a) For $\phi(t) = t$, show that no solution exists.
- (b) For $\phi(t) = 1$, show that there are infinitely many solutions.

Homework #2: Solutions

#2.25

Solve the following initial value problem:

$$U_t + 2U_x = \sin(x)$$

$$U(0, x) = \sin(x).$$

Solution:

Let $z = x - 2t$, $\tau = t$. Then,

$$\frac{\partial}{\partial t} = \frac{\partial z}{\partial t} \frac{\partial}{\partial z} + \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} = -2 \frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}$$

$$\Rightarrow -2U_z + U_\tau + 2U_z = \sin(z + 2\tau)$$

$$\Rightarrow U_\tau = \sin(z + 2\tau)$$

$$\Rightarrow U(\tau, z) = -\frac{\cos(z + 2\tau)}{2} + f(z)$$

$$\Rightarrow U(t, x) = -\frac{\cos(x)}{2} + f(x - 2t)$$

The initial conditions imply that $f(x) = \cos(x)/2 + \sin(x)$. Therefore,

$$U(t, x) = -\frac{\cos(x)}{2} + \frac{\cos(x - 2t)}{2} + \sin(x - 2t).$$

#2.28

Let $c \neq 0$. Prove that if the initial data satisfies $U(0, x) = V(x) \rightarrow 0$ as $x \rightarrow \pm\infty$, then, for each fixed x , the solution to the equation

$U_t + cU_x = 0$ satisfies $U(t, x) \rightarrow 0$ as $t \rightarrow \infty$.

Solution:

The solution to the equation

$$U_t + cU_x = 0,$$

$$U(0, x) = V(x)$$

is given by $U(t, x) = V(x - ct)$. Therefore, for fixed x it follows that

$$\lim_{t \rightarrow \infty} U(t, x) = \lim_{t \rightarrow \infty} V(x - ct) = \lim_{t \rightarrow -\infty} V(t) = 0.$$

#2.20

Consider the equation

$$u_t + (1+x^2)u_x = 0$$

- Find and sketch the characteristic curves.
- Write down a formula for the general solution.
- Find the general solution to the initial value problem $u(0, x) = f(x)$ and discuss its behavior as t increases.

Solution:

The characteristic curves satisfy:

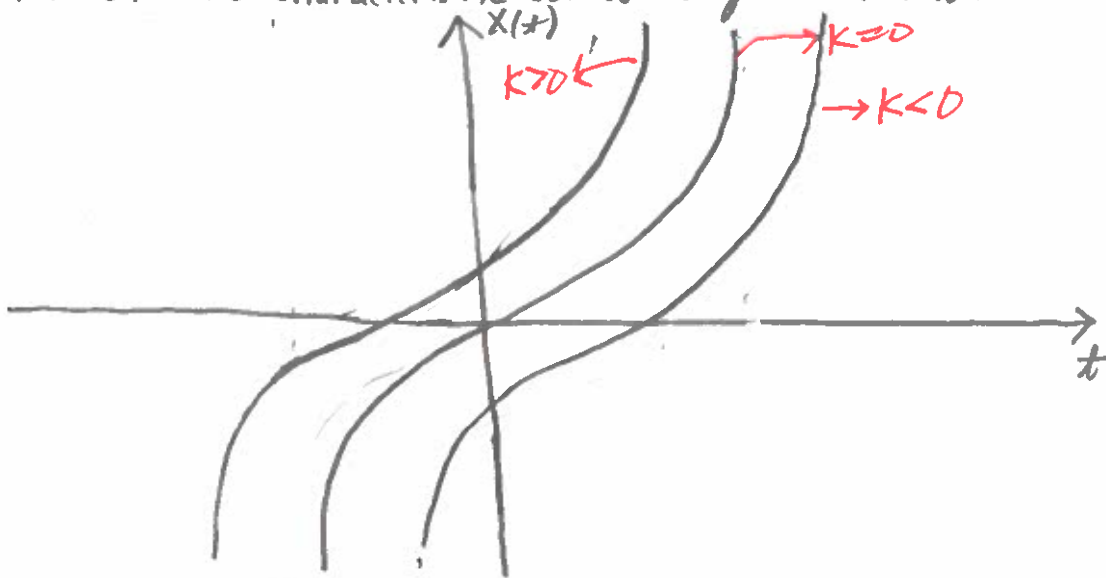
$$\frac{dx}{dt} = 1+x^2$$

$$\Rightarrow \int \frac{1}{1+x^2} dx = \int dt$$

$$\Rightarrow \tan^{-1}(x) = t + K$$

$$\Rightarrow x(t) = \tan(t + K)$$

The characteristic curves thus have singularities when $t + K = \pm \pi/2$.
The plot of the characteristic curves is given below:



It follows that $u(t, x)$ is constant along the curves $x(t) = \tan(t + K)$ and thus if we let $\xi = \tan^{-1}(x) - t$ then

$$u(t, x) = h(\xi) = h(\tan^{-1}(x) - t).$$

Therefore,

$$u(0, x) = h(\tan^{-1}(x)) = f(x)$$

$$\Rightarrow h(x) = f(\tan(x))$$

$$\Rightarrow u(t, x) = h(\xi) = f(\tan(\xi)) = f(\tan(\tan^{-1}(x) - t))$$

$$\Rightarrow u(t, x) = f\left(\frac{x - \tan(t)}{1 + x \tan(t)}\right).$$

Therefore, for each fixed value of x , the solution will have a singularity at

$$1+x+\tan(x)=0$$
$$\Rightarrow x = \tan^{-1}(-1-x).$$

Consequently, depending on the initial data the existence of a solution for all time is not guaranteed.

#4,

Consider the problem

$$\frac{d^2v}{dx^2} + v = 0$$

$$v(0) = 0 \text{ and } v(L) = 0$$

Where $L > 0$ is a constant and v is a function of a single variable x . Clearly $v(x) = 0$ is a solution. Is this solution unique or not.

Solution:

The generic solution to this equation is of the form

$$v(x) = A \cos(x) + B \sin(x)$$

$$v(0) = 0 \Rightarrow A = 0$$

$$v(L) = 0 \Rightarrow B \sin(L) = 0.$$

Therefore, $v = 0$ is the unique solution if $L \neq \pi$.

#5

Solve the boundary value problem

$$v''(x) = 0$$

$$v'(0) + kv(0) = 0$$

$$v'(l) \pm kv(l) = 0.$$

Solution:

The generic solution is of the form:

$$v(x) = ax + b.$$

The boundary conditions imply

$$a + kb = 0$$

$$a \pm k(a+b) = 0$$

+ Case:

$$a + kb = 0$$

$$(1+k)a + kb = 0$$

$$\Rightarrow a + ka - a = 0$$

Therefore, if $k \neq 0$ then $a=0$ and $b=0$. If $k=0$, then $a=0$ and b is a free variable.

- Case:

$$\begin{aligned} a+kb &= 0 \\ (1-k)a+kb &= 0 \\ \Rightarrow a-ka+a &= 0 \\ \Rightarrow a(2-k) &= 0 \end{aligned}$$

Therefore, if $k \neq 2$, $a=0$ and $b=0$. If $k=2$, then $a=-2b$ and b is a free variable.

#2.2.19

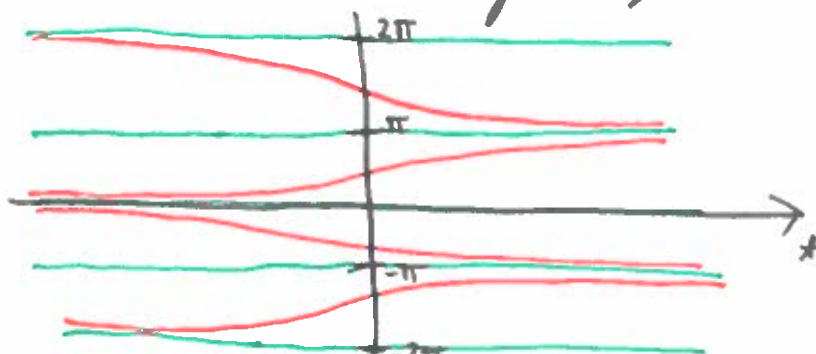
- Find and graph the characteristic curves for the equation $u_t + \sin(x)u_x = 0$.
- Write down the solution with initial data $u(0,x) = \cos(\frac{1}{2}x)$.
- Graph your solution at times $t=0, 1, 2, 3, 5, 10$.
- What is the limiting profile as $t \rightarrow \infty$.

Solution:

a.) The characteristic curves satisfy

$$\begin{aligned} \frac{dx}{dt} &= \sin(x) \\ \Rightarrow -\ln(\cot(x) + \csc(x)) &= t + K \\ \Rightarrow -\ln\left(\cot\left(\frac{x}{2}\right)\right) &= t + K \\ \Rightarrow \cot\left(\frac{x}{2}\right) &= Ce^{-t} \\ \Rightarrow \tan\left(\frac{x}{2}\right) &= Ce^t \\ \Rightarrow x &= 2 + \tan^{-1}(Ce^t) \end{aligned}$$

The characteristic curves are given by:



b.) Since $u(t,x)$ is constant along the characteristic curves it follows that

$$u(t,x) = h(-\ln(\cot(\frac{x}{2})) - t)$$

$$\Rightarrow u(0,x) = h(\ln(\tan(\frac{x}{2})) = f(x)$$

$$\Rightarrow h(x) = f(2 \tan^{-1}(e^x)) = \cos(\pi \tan^{-1}(e^x))$$

$$\begin{aligned} \Rightarrow u(t,x) &= \cos(\pi \tan^{-1}(e^{\ln(\tan(\frac{x}{2})) - t})) \\ &= \cos(\pi \tan^{-1}(\tan(\frac{x}{2}) - t) \end{aligned}$$

d.) The limiting profile is easiest to calculate from the characteristic curves.

$$\lim_{t \rightarrow \infty} u(t,x) = \begin{cases} 1 & \text{if } x \in (0, \pi) \cup (-\pi, 0) \cup \dots \\ -1 & \text{if } x \in (\pi, 2\pi) \cup (-2\pi, -\pi) \cup \dots \\ 1 & \text{if } x = 0, 4\pi, -4\pi, 8\pi, -8\pi, 12\pi, -12\pi, \dots \\ -1 & \text{if } x = 2\pi, -2\pi, 6\pi, -6\pi, \dots \\ 0 & \text{if } x = \pi, -\pi, 3\pi, -3\pi, 5\pi, -5\pi, \dots \end{cases}$$

#2.2.25

a.) Suppose $c(x) \in C^1$. Prove that characteristic curves of the transport equation cannot cross each other.

Proof:

The characteristic curves satisfy

$$\frac{dx}{dt} = c(x).$$

By existence and uniqueness, the characteristic curves cannot intersect.

G-e) Prove that characteristic curves are monotone.

Proof:

Let $g(t)$ be a characteristic curve. Then,

$$\frac{d}{dt} g(t+\delta) = g'(t+\delta) = c(g(t+\delta)).$$

Therefore, characteristics must be monotone else they would intersect. ■

#2.

Consider the equation

$$u_x + x u_x = 0$$

with the boundary condition $u(x, 0) = \phi(x)$.

a.) For $\phi(x) = x$, no solution exists.

b.) For $\phi(x) = 1$, show there are infinitely many solutions.

Solution:

a.) Since the characteristics are $x(t) = ke^t$ it follows that along the curve $x=0$, $u(x, t)$ must be constant.

b.) There are infinitely many solutions for $\phi(x) = 1$ since there are infinitely many functions that are constant along characteristics and satisfy $u(x, 0) = 1$.