MST 352/652
Homework #3

Due Date: February 05, 2019

1 Problems for everyone

1. Consider the following initial value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0,$$
$$u(0,x) = \frac{4}{4+x^2} \text{ and } u_t(0,x) = 0,$$

where $c > 0$ is a constant.

(a) What do the initial conditions model in this situation?

(b) Calculate the exact solution $u(x,t)$ to this problem.

(c) Assuming $c = 1$, plot $u(x,t)$ at $t = 0$, $t = 1$, and $t = 2$.

(d) Calculate $\lim_{t\to\infty} u(t,x)$.

(e) Discuss the qualitative behavior of the solution.

2. Consider the following initial value problem for the wave equation:

$$u_{tt} = c^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0,$$
$$u(0,x) = 0 \text{ and } u_t(0,x) = \frac{4}{4+x^2},$$

where $c > 0$ is a constant.

(a) What do the initial conditions model in this situation?

(b) Calculate the exact solution $u(t,x)$ to this problem.

(c) Assuming $c = 1$, plot $u(t,x)$ at $t = 0$, $t = 1$, and $t = 2$.

(d) Calculate $\lim_{t\to\infty} u(t,x)$.

(e) Contrast how the solution to this initial value problem is different from the solution to problem #1.

3. pg. 60-62, #2.4.1-#2.4.3

4. For a classical solution $u(t,x)$ of the wave equation $u_{tt} = u_{xx}$ the energy and momentum density are defined by

$$E(t,x) = \frac{1}{2}\left(u_t^2 + u_x^2\right) \text{ and } P(t,x) = u_t u_x,$$

respectively.

(a) Show that $E_t = P_x$ and $E_x = P_t$.

(b) Show that both $E(t,x)$ and $P(t,x)$ satisfy the wave equation.

5. If $u(t,x)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the following identity

$$u(x+h,t+k) + u(x-h,t-k) = u(x+k,t+h) + u(x-k,t-h),$$

for all $x, t, h, k \in \mathbb{R}$.

6. In 1858 the first trans-Atlantic telegraph cable was laid down allowing direct communication between the United States and England. The first message sent by Queen Victoria to President James Buchanan contained 98 words and took 16 hours to communicate. This was an incredible improvement over the ten days it took to communicate by ships! However, within a month of its first operation Dr. Wildman Whitehouse (a medical doctor with no real training in electrical engineering) essentially melted the cable by applying too high of a voltage. The reaction from the media to this announcement was volatile with many writers hinting that the line was a hoax. Nevertheless, engineers persisted and a working cable was installed in 1866 that could transmit a blazing eight words per minute!

In this problem our goal is to mathematically understand how signals can be sent along wires. In particular, our goal is to tune specific electrical properties of the wire to arrive at the wave equation. As we saw in class, the wave equation is particularly nice for sending signals because the solutions retain the shape of the initial profile. That is, somebody receiving the signal would know precisely what the initial shape of the signal is.

(a) We will model the wire as an infinitely long straight line of resistance $R \geq 0$, inductance $L \geq 0$, grounding resistance $G \geq 0$ and capacitance $C \geq 0$. Letting $I(t,x)$ and $V(t,x)$ denote the current and voltage across the wire at position x and time t, the partial differential equations satisfied by I and V are given by:

$$V_x = -RI - LI_t, \tag{1}$$
$$I_x = -GV - CV_t. \tag{2}$$

Show that I can be eliminated from these equations to yield the following equation:

$$LCV_{tt} + (LG + RC)V_t + RGV = V_{xx}. \tag{3}$$

(b) If we define $c = 1/\sqrt{LC}$, $a = c^2(LG + RC)$, and $b = c^2RG$ then the above equation becomes:

$$V_{tt} + aV_t + bV = c^2V_{xx}. \tag{4}$$

This equation is known as the **telegrapher's equation**. If $a = 0$ and $b = 0$ this equation becomes the wave equation. If $L > 0$ and $C > 0$ show that if $a = 0$ and $b = 0$ then both $R = 0$ and $G = 0$. Why is this an unrealistic case?

(c) By making a substitution of the form $V(t,x) = e^{-\lambda t}u(t,x)$ and by appropriately picking λ in terms of the constants a, b and c, show that the telegraphers equation can be reduced to the following partial differential equation:

$$u_{tt} + ku = c^2u_{xx}, \tag{5}$$

where k is a constant that depends on a, b and c.

(d) Show that $k = 0$ only when $RC = LG$.

(e) In the case when $RC = LG$ with the initial conditions $u(0, x) = f(x)$ and $u_t(0, t) = 0$, show that a solution to a solution to the telegrapher's equation is given by

$$V(t, x) = \frac{e^{-\lambda t}}{2}(f(x - ct) + f(x + ct)), \tag{6}$$

What does this solution means in practical terms? In particular, why is it useful for sending signals?

Remark: This process of tuning the electrical parameters so the $RC = LG$ is called "Pupinizing" the cable after one of its discoverers Michael Pupin.

2 Graduate Problems

1. A spherical wave is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance from the origin. The wave equation in this case takes the form:

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right)$$
$$u(0, r) = \phi(r),$$
$$u_t(0, r) = \psi(r)$$

where $\phi(r), \psi(r)$ are both even functions of r. By making the change of variables $v = ru$, solve the spherical wave equation.

2. #2.4.15

Solutions

#2.4.2.

a.) Solve the wave equation $u_{tt}=u_{xx}$ when the initial displacement is the box function

$$u(0,x)=\begin{cases} 1, & 1<x<2 \\ 0, & \text{o.w.} \end{cases}$$

b.) Sketch the resulting solution at several representative times.

Solution:

a.) Let $y=x-3/2$. The equation becomes

$$u_{tt}=u_{yy}$$

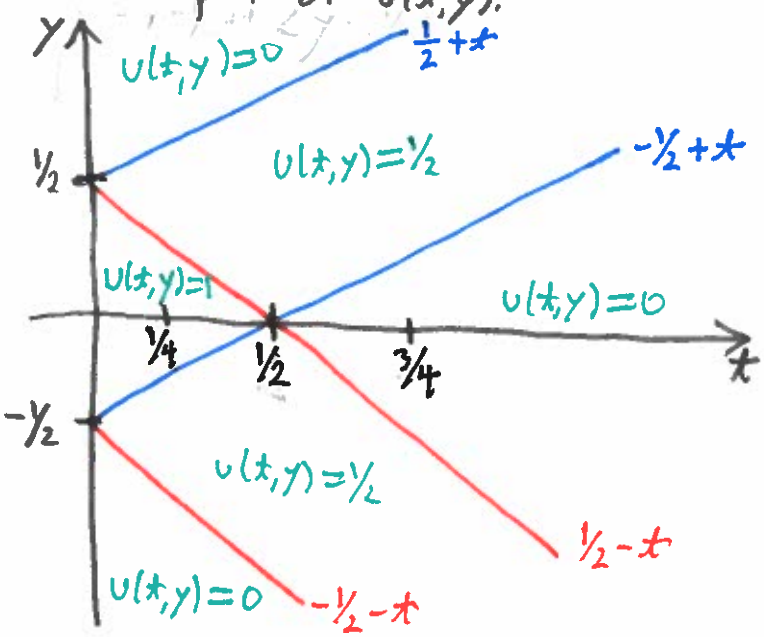
$$u(0,y)=\begin{cases} 1, & -1/2<y<1/2 \\ 0, & \text{o.w.} \end{cases} = f(y)$$

$$\Rightarrow u(t,y)=\frac{f(y-t)}{2}+\frac{f(y+t)}{2}.$$

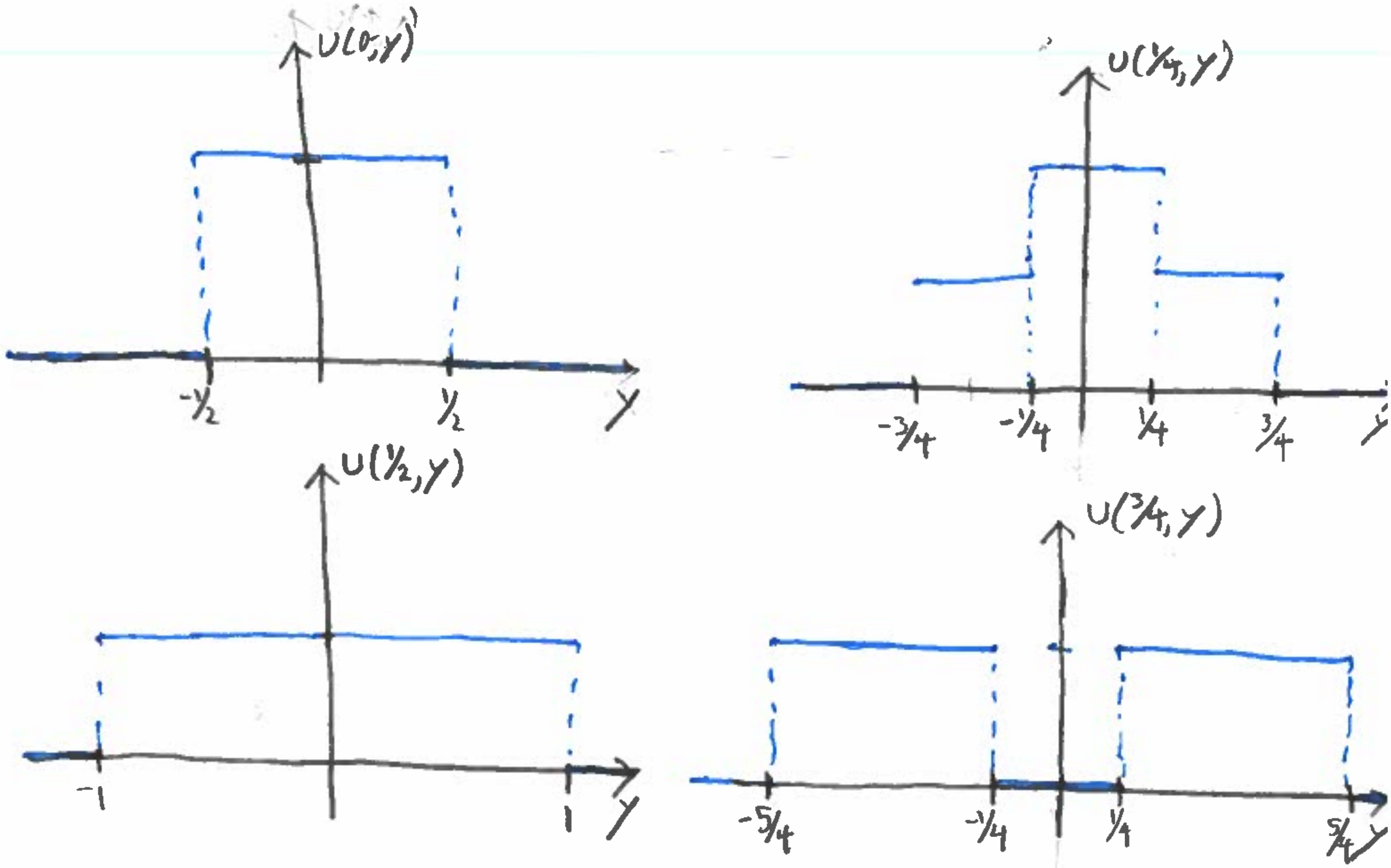
Explicitly,

$$u(t,y)=\begin{cases} 1/2, & -1/2+t<y<1/2+t \\ 0, & \text{o.w.} \end{cases} + \begin{cases} 1/2, & -1/2-t<y<1/2-t \\ 0, & \text{o.w.} \end{cases}.$$

Finding all the conditions is hard. It is easier to think graphically. Below is a contour plot of $u(t,y)$.



b.) We can use the contour plot to graph the solution at different times.



#2.4.2

Answer Exercise #2.4.2 when the initial velocity is the box function.

Solution:

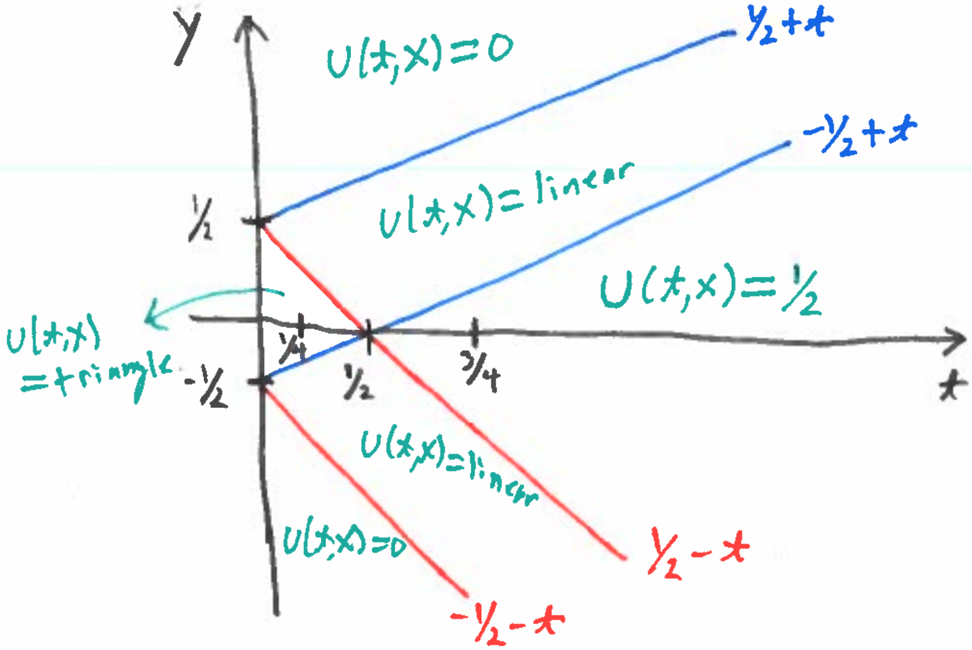
Again let $y = x - 3/2$. The solution is then given by:

$$u(t,y) = \frac{1}{2}\int_{y-t}^{y+t} f(s)\,ds$$

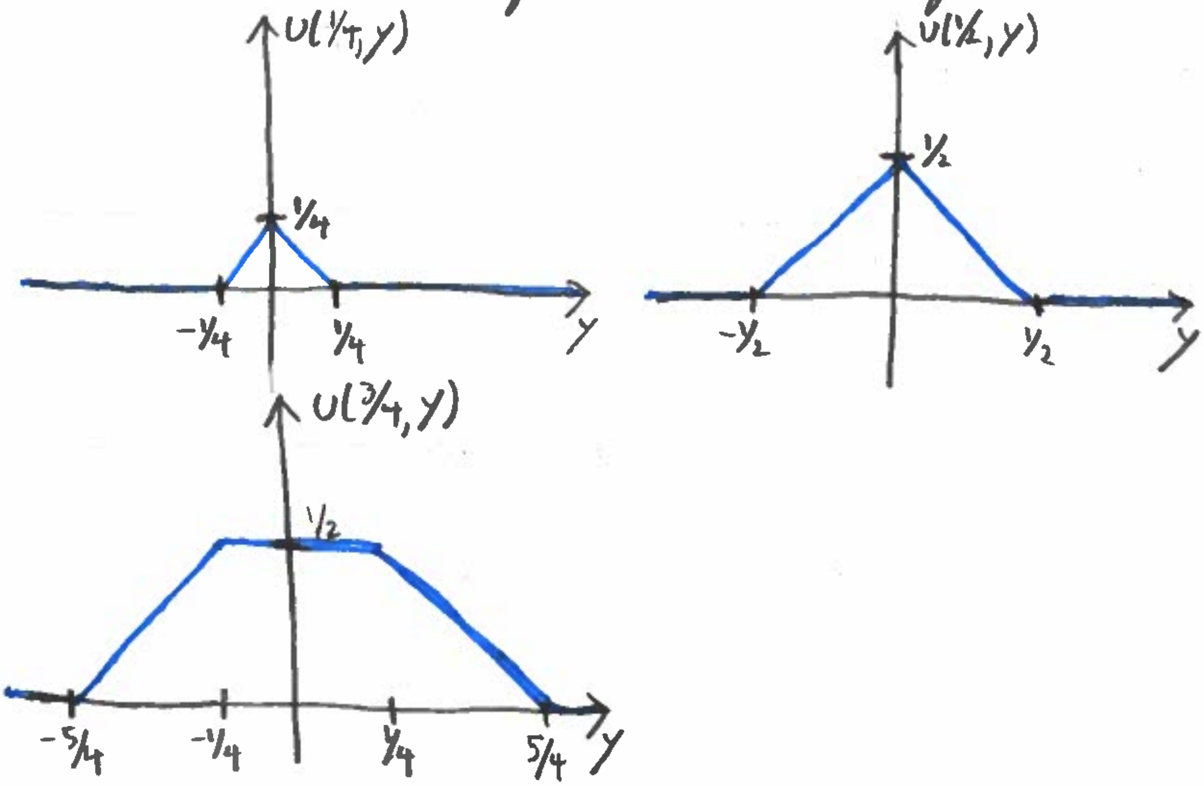
The exact formula is difficult to compute directly. It is easier to think graphically. We first make the following observations:

1. If $y < 1/2 + t$ and $y > 1/2 - t$ then $u(t,y) = 1/2$
2. If $y < -1/2 - t$ or $y > 1/2 + t$ then $u(t,y) = 0$.
3. In other regions, $u(t,y)$ is linear.

This allows us to draw a "phase" diagram which illustrates the behavior of the solution.



We can use this diagram to draw the figure.



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#4.

For a classical solution of the wave equation $u_{tt}=u_{xx}$ the energy and momentum density are defined by

$$E(t,x)=\frac{1}{2}(u_t^2+u_x^2) \text{ and } P(t,x)=u_t u_x,$$

respectively.

a.) Show that $E_t=P_x$ and $E_x=P_t$.

b.) Show that both E and P satisfy the wave equation.

Solution:

a.) $E_t=u_t u_{tt}+u_x u_{xt}=u_t u_{xx}+u_x u_{xt}$

$E_x=u_t u_{tx}+u_x u_{xx}=u_t u_{tx}+u_x u_{tt}$

$P_t=u_{tt}u_x+u_t u_{xt}$

$P_x=u_{tx}u_x+u_t u_{xx}$

b.) By part (a) it follows that

$E_{tt}=P_{xt}=P_{tx}=E_{xx}$

$P_{tt}=E_{xt}=E_{tx}=P_{xx}$.

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#5.

If $u(t,x)$ satisfies the wave equation $u_{tt}=u_{xx}$, prove the following identity

$$u(t+h,x+k)+u(t-h,x-k)=u(t+k,x+h)+u(t-k,x-h).$$

Solution:

Given that $u(t,x)$ satisfies $u_{tt}=u_{xx}$, it follows that there exists functions p and q such that

$$u(t,x)=p(x-t)+q(x+t).$$

Therefore,

$$\begin{aligned} u(t+h,x+k)+u(t-h,x-k) &= p(x+k-t-h)+q(x+k+t+h)+p(x-k-t+h)+q(x-k+t-h) \\ &= p(x+h-(t+k))+q(x+h+t+k)+p(x-h-(t-k))+q(x-h+t-k) \\ &= u(t+k,x+h)+u(t-k,x-h) \end{aligned}$$

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#6.

a.) Consider the current and voltage equations for a wire:

$$V_x = -RI - LI_t$$
$$I_x = -GV - CV_t.$$

Show that I can be eliminated from these equations to yield:

$$LCV_{tt} + (LG+RC)V_t + RGV = V_{xx}$$

Solution:

Differentiating it follows that

$$V_{xx} = -RI_x - LI_{tx}$$
$$I_{tx} = -GV_t - CV_{tt}$$

$$\Rightarrow V_{xx} = -R(-GV - CV_t) - L(-GV_t - CV_{tt})$$
$$\Rightarrow V_{xx} = LCV_{tt} + (LG+RC)V_t + RGV.$$

C.) By making a substitution of the form $V(t,x) = e^{-\lambda t}u(t,x)$, show that the telegraphers equation can be reduced to:

$$u_{tt} + Ku = c^2 u_{xx}.$$

Solution:

Let $V(t,x) = e^{-\lambda t}u(t,x)$. Then,

$$V_t = -\lambda e^{-\lambda t}u + e^{-\lambda t}u_t$$
$$V_{tt} = \lambda^2 e^{-\lambda t}u - 2\lambda e^{-\lambda t}u_t + e^{-\lambda t}u_{tt}$$
$$V_x = e^{-\lambda t}u_x$$
$$V_{xx} = e^{-\lambda t}u_{xx}$$

The telegraphers equation becomes

$$u_{xx} = LC(\lambda^2 u - 2\lambda u_t + u_{tt}) + (LG+RC)(-\lambda u + u_t) + RGu$$
$$\Rightarrow u_{xx} = (LC\lambda^2 - \lambda(RC+LG) + RG)u + (LG+RC-2\lambda LC)u_t + LCu_{tt}$$

Choose $\lambda = (LG+RC)/2LC$. Therefore,

$$u_{xx} = \left(\frac{(LG+RC)^2}{4LC} - \frac{(RC+LG)^2}{2LC} + RG\right)u + LCu_{tt}$$

$$\Rightarrow \frac{1}{LC}u_{xx} = \left(RG - \frac{(LG+RC)^2}{2LC}\right)u + u_{tt}$$

$$\Rightarrow \frac{1}{LC}u_{xx} = -\frac{(LG-RC)^2}{2LC}u + u_{tt}$$

e.) In the case $RC=LG$, with the initial condition $U(0,x)=f(x)$ and $U_t(0,t)$, show that a solution is given by

$$V(t,x)=\frac{e^{-\lambda t}}{2}\left(f(x-ct)+f(x+ct)\right)$$

Solution:

When $RC=LG$ we obtain the wave equation

$$U_{tt}=c^2U_{xx}$$

$$\Rightarrow V(t,x)=\frac{e^{-\lambda t}}{2}\left(f(x-ct)+f(x+ct)\right)$$

$$=\frac{1}{2}\exp\left(-\frac{(LG+RC)}{2LC}t\right)\left(f\left(x-\frac{t}{\sqrt{LC}}\right)+f\left(x+\frac{t}{\sqrt{LC}}\right)\right).$$

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Graduate Problems

#1

A spherical wave is a solution of the three dimensional wave equation of the form $u(r,t)$, where r is the distance from the origin. The wave equation in this case takes the form:

$$U_{tt}=c^2\left(U_{rr}+\frac{2}{r}U_r\right)$$

$$U(0,r)=\phi(r)$$

$$U(0,r)=\psi(r)$$

Where $\phi(r)$, $\psi(r)$ are both even functions of r. By making the change of variables $V=rU$, solve the spherical wave equation.

Solution:

Let $V=rU$. Then, $U=V/r$ and it follows from differentiating that

$$U_r=\frac{V_r}{r}-\frac{V}{r^2}$$

$$U_{rr}=\frac{V_{rr}}{r}-\frac{2V_r}{r^2}+\frac{2V}{r^3},$$

$$U_{tt}=\frac{V_{tt}}{r}.$$

Therefore,

$$\frac{V_{tt}}{r}=c^2\left(\frac{V_{rr}}{r}-\frac{2V_r}{r^2}+\frac{2V}{r^3}+\frac{2V_r}{r^2}-\frac{2V}{r^3}\right)$$

$$\Rightarrow V_{tt}=c^2V_{rr}.$$

Therefore, we have the following initial value problem:

$$V_{tt} = c^2 V_{xx}$$
$$V(0,r) = r\phi(r)$$
$$V_t(0,r) = r\psi(r)$$

The unique solution is therefore given by:

$$V(t,r) = \frac{1}{2}(r-ct)\phi(r-ct) + \frac{1}{2}(r+ct)\phi(r+ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} s\,\psi(s)\,ds$$

#2.4.15

Prove uniqueness for the following initial value problem:

$$(*) \quad \begin{aligned} &u_{tt} + a u_t = c^2 u_{xx} \\ &u(0,x) = f(x) \\ &u_t(0,x) = g(x). \end{aligned}$$

Solution:

Let $E[u](t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}c^2 u_x^2 + \frac{1}{2}u_t^2\right)dx$. Therefore, if u satisfies $(*)$ it follows that

$$\begin{aligned}\frac{dE}{dt} &= \int_{-\infty}^{\infty}\left(c^2 u_x u_{xt} + u_t u_{tt}\right)dx \\ &= c^2 u_x u_t\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} c^2 u_{xx} u_t\,dx + \int_{-\infty}^{\infty} u_t u_{tt}\,dx \\ &= -\int_{-\infty}^{\infty}\left(-u_{tt}u_t - a u_t^2 + u_t u_{tt}\right)dx \\ &= -\int_{-\infty}^{\infty} a u_t^2\,dx \\ &\le 0.\end{aligned}$$

Therefore, E is monotone decreasing. Suppose u_1, u_2 solve $*$. Then $v = u_1 - u_2$ solves

$$v_{tt} + a v_t = c^2 v_{xx}$$
$$v(0,x) = 0$$
$$v_t(0,x) = 0$$

Consequently, $E[v](0) = 0$ and thus for all t, $E[v](t) = 0$. Consequently, for all (t,x) it follows that $v_t(t,x) = v_x(t,x) = 0$. Therefore $v = 0$ proving that $u_1 = u_2$.