MST 352/652
Homework #4

Due Date: February 12, 2019

1 Problems for everyone

1. pg. 72, #3.1.2, #3.1.4-3.1.6.

2. Consider the following initial boundary value problem on the domain $[0, \pi]$:

$$\begin{cases} tu_t = u_{xx} + 2u \\ u(t,0) = u(t,\pi) \\ u(0,x) = 0 \end{cases} .$$

   By separating variables, show that this initial boundary value problem has an infinite number of solutions.

3. Let $l, k > 0$. Find all separable eigensolutions of the Schrödinger equation $u_t = iku_{xx}$ on the interval $[0, l]$ subject to following boundary conditions:

$$u_x(t,0) = 0 \text{ and } u(t,l) = 0.$$

4. The overdamped dynamics of an elastic beam on the interval $[0, l]$, i.e. a thin plank of wood of length l, can be modeled by the following initial boundary value problem:

$$\begin{cases} u_t = u_{xxxx} \\ u(0,x) = f(x) \\ u(t,0) = u(t,l) = 0 \\ u_{xx}(t,0) = u_{xx}(t,l) = 0 \end{cases}.$$

   (a) Find all separable eigensolutions to this equation.
   (b) What does the initial condition model?
   (c) What do the boundary conditions model?

5. pg. 76, #3.2.1-3.2.4.

6. Let f be a periodic function of period p.

   (a) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x)\,dx = \int_a^{a+p} f(x)\,dx.$$

   Hint: Write $\int_a^{a+p} f(x)\,dx$ as the sum of two integrals (a to p, and p to $a+p$) and make an appropriate change of variables.

(b) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x+a)\,dx = \int_0^p f(x)\,dx.$$

(c) Interpret these identities graphically.

2 Graduate Problems

1. #3.1.7

2. Let $l > 0$. Consider the wave equation on the domain $[0, l]$ with so called Robin boundary conditions:

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \\ u_x(t,0) = u(t,0) \\ u_x(t,l) = u(t,l) \end{cases}.$$

   By constructing an appropriate energy, prove that solutions to this initial boundary value problem are unique.

   Hint: The energy I defined in class won't work, but it might provide hints on how to add additional terms that yield an energy that is conserved for all time.

3. #3.2.10

Homework #4

#3.1.2.

Find all seperable eigensolutions to the heat equation $u_t = u_{xx}$ on the interval $0 \le x \le \pi$ subject to the boundary conditions:

a.) $u(t,0)=0$, $u(t,\pi)=0$

b.) $u(t,0)=0$, $u_x(t,\pi)=0$

c.) $u_x(t,\pi)=0$, $u_x(t,\pi)=0$

Solution:

Seperable solutions to the heat equation satisfy the following eigenvalue problems:

$$X''=\lambda X \quad \text{and} \quad T'=\lambda T.$$

We split the solutions into three cases:

Case 1 ($\lambda>0$):
$$X = Ae^{\sqrt{\lambda}x}+Be^{-\sqrt{\lambda}x}$$

Case 2 ($\lambda=0$):
$$X = ax+b$$

Case 3 ($\lambda<0$):
$$X = A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$$

a.) For the boundary conditions $u(t,0)=u(t,\pi)=0$ we check each case:

Case 1 ($\lambda>0$):

$$u(t,0)=0 \Rightarrow A=-B.$$

Consequently,

$$X = A(e^{\sqrt{\lambda}x}-e^{-\sqrt{\lambda}x}) = 2A\sinh(\sqrt{\lambda}x)$$
$$\Rightarrow X(\pi)=2A\sinh(\sqrt{\lambda}\pi)=0 \Rightarrow A=0.$$

Case 2 ($\lambda=0$):

$$u(t,0)=b=0$$
$$u(t,\pi)=a\pi=0$$
$$\Rightarrow a=0.$$

Case 3 ($\lambda<0$):

$$u(t,0)=A=0$$
$$u(t,\pi)=B\sin(\sqrt{\lambda}\pi)e^{\lambda t}=0$$

The eigenfunctions are thus given by:

$$X_n(x)=B\sin(nx).$$

b.) For the boundary conditions $u(t,0)=u_x(t,\pi)=0$ we check cases:

Case 1 $(\lambda>0)$:

$u(t,0)=0 \Rightarrow A=-B$.

Consequently, $X(x)=2A\sinh(\sqrt{\lambda}x)$.

$$\Rightarrow u_x(t,\pi)=2A\sqrt{\lambda}\cosh(\sqrt{\lambda}\pi)e^{\lambda t} \Rightarrow A=0$$

Case 2 $(\lambda=0)$:

$X(x)=ax+b$

$u(x,0)=0 \Rightarrow b=0$

$u_x(x,\pi)=0 \Rightarrow a=0$

Case 3 $(\lambda<0)$:

$u(t,0)=0 \Rightarrow A=0$

$\Rightarrow u_x(t,x)=B\sqrt{\lambda}\cos(\sqrt{\lambda}x)e^{-\lambda t}$

$\Rightarrow u_x(t,\pi)=B\sqrt{\lambda}\cos(\sqrt{\lambda}\pi)e^{-\lambda t}$

Therefore, the eigenfunctions are given by:

$$X_n(x)=\sin\left(\frac{(2n-1)\pi}{2}x\right),$$

With eigenvalues $\lambda_n=(2n-1)^2/4$.

c.) For the boundary conditions $u_x(t,0)=u_x(t,\pi)=0$ we check cases:

Case 1 $(\lambda>0)$:

$X(x)=A\exp(\sqrt{\lambda}x)+B\exp(-\sqrt{\lambda}x)$

$u_x(t,0)=0 \Rightarrow A=B$

$\Rightarrow X(x)=2A\cosh(\sqrt{\lambda}x)$

$\Rightarrow u_x(t,\pi)=2A\sqrt{\lambda}\sinh(\sqrt{\lambda}\pi)e^{-\lambda t}=0$

$\Rightarrow A=0$

Case 2 $(\lambda=0)$:

$X(x)=ax+b$

$\Rightarrow u_x(t,0)=u_x(t,\pi)=0 \Rightarrow u(t,x)=b.$

Case 3 $(\lambda<0)$:

$X(x)=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$

$u_x(t,0)=0 \Rightarrow B=0$

$u_x(t,\pi)=-\sqrt{\lambda}A\sin(\sqrt{\lambda}\pi)e^{-\lambda t}=0$

The eigenfunctions are thus given by:

$$X_n(x)=\cos(nx);\quad X_0(x)=1.$$

#3.1.4.

Find all seperable eigensolutions to the following partial differential equations:

a.) $U_t = U_x$

b.) $U_t = U_x - U$

c.) $U_t = x U_x$

Solution:

Assume $u(t,x) = T(t)X(x)$. Then

a.) $T'X = TX'$

$\Rightarrow T' = \lambda T$ and $X' = \lambda X$

$\Rightarrow T = e^{\lambda t}$ and $X = Ae^{\lambda x}$.

b.) $T'X = TX' - TX$

$\Rightarrow T' = \lambda T$ and $X' - X = \lambda X$

$\Rightarrow T = e^{\lambda t}$ and $X = Ae^{(1+\lambda)t}$.

c.) $T'X = xTX'$

$\Rightarrow T' = \lambda T$ and $xX' = \lambda X$

$\Rightarrow T = e^{\lambda t}$ and $X = Ax^{\lambda}$

#2.

Consider the following initial boundary value problem on $[0,\pi]$:

$$tU_t = U_{xx} + 2U$$
$$U(t,0) = U(t,\pi)$$
$$U(0,x) = 0$$

Solution:

Let $u(t,x) = TX$

$\Rightarrow tT'X = TX'' + 2TX$

$\Rightarrow \frac{tT'}{T} = \frac{X'' + 2X}{X} = \lambda$

$\Rightarrow T' = \frac{\lambda}{t}T, \quad X'' = (\lambda - 2)X$

$\Rightarrow T = t^{\lambda}$ and $X = A\exp(\sqrt{\lambda-2}\,x) + B\exp(-\sqrt{\lambda-2}\,x)$

If $\lambda = 2$ we obtain the infinite family of solutions:

$u(t,x) = At^2$.

#4.

The overdamped dynamics of an elastic beam on an interval $[0,\ell]$ can be modeled by the following initial boundary value problem:

$$\begin{cases} U_t = U_{xxxx} \\ u(0,x) = f(x) \\ u(t,0) = u(t,\ell) = 0 \\ U_{xx}(t,0) = U_{xx}(t,\ell) = 0 \end{cases}$$

a.) Find all seperable eigensolutions to this equation.
b.) What does this initial Condition model?
c.) What do the boundary conditions model?

Solution:

a.) Let $u(t,x) = T\mathbb{X}$. Therefore,

$$\frac{T'}{T} = \frac{\mathbb{X}^{(IV)}}{\mathbb{X}} = \lambda$$

$$\Rightarrow \mathbb{X} = A\exp(\lambda^{1/4}x) + B\exp(-\lambda^{1/4}x) + C\exp(\lambda^{1/4}ix) + D\exp(-i\lambda^{1/4}x)$$

$$= A\exp(\lambda^{1/4}x) + B\exp(-\lambda^{1/4}x) + C\cos(\lambda^{1/4}x) + D\sin(\lambda^{1/4}x)$$

$$\mathbb{X}(0) = 0 \Rightarrow A + B + C = 0$$

$$\mathbb{X}''(0) = 0 \Rightarrow A\lambda^{1/2} + B\lambda^{1/2} - C\lambda^{1/2} = 0$$

$$\Rightarrow A + B = -C$$
$$A + B = C$$

One solution is $A = B = C = 0$. The generic eigenfunctions are thus given by

$$\mathbb{X} = A\sin\left(\frac{n\pi x}{\ell}\right)$$

With the temporal term given by

$$T(t) = \exp\left(\frac{n^4\pi^4}{\ell^4}t\right)$$

b.) The initial condition models the initial displacement of the plank.

c.) The boundary conditions model that the beam is clamped. ■

#3.2.1

Find the Fourier series of the following functions

a.) $\text{sign}(x)$

b.) $|x|$

c.) $3x-1$

d.) x^2

Solution:

a.) $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx$$
$$= -\frac{2}{n\pi}\cos(nx)\Big|_0^{\pi}$$
$$= \frac{-2}{n\pi}\left((-1)^n - 1\right)$$
$$= \frac{2}{n\pi}\left((-1)^{n+1} + 1\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin((2n-1)x)$$

b.) $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$
$$= \frac{2}{\pi n}\left(x\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\sin(nx)\,dx\right)$$
$$= \frac{-2}{\pi n^2}\cos(nx)\Big|_0^{\pi}$$
$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos((2n-1)x)$$

c.) $f(x) = 3x - 1$

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(-1)\,dx = -2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(-1)\cos(nx)\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} 3x\sin(nx)\,dx$$

$$= \frac{6}{\pi}\left(-\frac{x\cos(nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\,dx\right.$$

$$= \frac{6}{n}(-1)^{n+1}$$

$$\Rightarrow f(x) = 6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) - 1$$

d.) $f(x) = x^2$

Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{2x\sin(nx)}{n}\,dx\right)$$

$$= -\frac{4}{\pi n}\int_0^{\pi} x\sin(nx)\,dx$$

$$= -\frac{4}{\pi n^2}\,x\cos(nx)\Big|_0^{\pi}$$

$$= -\frac{4}{n^2}(-1)^n$$

$$= \frac{4}{n^2}(-1)^{n+1}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nx)$$

#6.

Let f be a periodic function of p.

a) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x)\,dx = \int_a^{a+p} f(x)\,dx$$

b) Prove that for any $a \in \mathbb{R}$:

$$\int_0^p f(x+a)\,dx = \int_0^p f(x)\,dx.$$

Solution:

$$
\begin{aligned}
\text{a.)}\ \int_a^{a+p} f(x)\,dx &= \int_a^p f(x)\,dx + \int_p^{a+p} f(x)\,dx \\
&= \int_a^p f(x)\,dx + \int_0^a f(u+p)\,du \\
&= \int_a^p f(x)\,dx + \int_0^a f(u)\,du \\
&= \int_0^p f(x)\,dx.
\end{aligned}
$$

$$\text{b.)}\ \int_0^p f(x+a)\,dx = \int_a^{p+a} f(u)\,du = \int_0^p f(u)\,du.$$

Graduate Problems

#2.

Let $l > 0$. Consider the wave equation on $[0,l]$ with Robin boundary conditions:

$$
\star\quad
\begin{aligned}
&u_{tt} = u_{xx} \\
&u(0,x) = f(x) \\
&u_t(0,x) = g(x) \\
&u_x(t,0) = u(t,0) \\
&u_x(t,l) = -u(t,l)
\end{aligned}
$$

Solution:

Let

$$E(t) = \frac{1}{2}u(t,0)^2 + \frac{1}{2}u(t,l)^2 + \int_0^l \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right)dx$$

$$
\begin{aligned}
\Rightarrow \frac{dE}{dt} &= u(t,0)u_t(t,0) + u(t,l)u_t(t,l) + \int_0^l (u_t u_{tt} + u_x u_{xt})\,dx \\
&= u(t,0)u_t(t,0) + u(t,l)u_t(t,l) + \int_0^l (u_t u_{xx} + u_x u_{xt})\,dx \\
&= u(t,0)u_t(t,0) + u(t,l)u_t(t,l) + u_t u_x\Big|_0^l + \int_0^l (-u_{tx}u_x + u_x u_{tx})\,dx
\end{aligned}
$$

$$\Rightarrow \frac{dE}{dt} = u(t,0)u_x(t,0) + u(t,\ell)u_x(t,\ell) - u(t,\ell)u_x(t,\ell) - u(t,0)u_x(t,\ell)$$
$$= 0.$$

Therefore, $E(t)$ is conserved.

Now, let u_1, u_2 solve ☆. Then, $w = u_1 - u_2$ satisfies:

$$w_{tt} = w_{xx}$$
$$w(0,x) = 0$$
$$w_t(0,x) = 0$$
$$w(t,0) = w_x(t,0)$$
$$w(t,\ell) = w_x(t,\ell)$$

Therefore, for all t it follows that

$$E(t) = 0$$
$$\Rightarrow \int_0^\ell \left(\frac{1}{2}w_t^2 + \frac{1}{2}w_x^2\right)dx + \frac{w(t,0)^2}{2} + \frac{w(t,\ell)^2}{2} = 0$$
$$\Rightarrow w = 0.$$

#3.2.10

Let f be a smooth 2π-periodic function.

a.) Prove that f' is 2π-periodic

b.) Show that if f has mean zero then
$$g(x) = \int_0^x f(s)ds$$
is 2π-periodic.

d.) Prove that if f has mean m, then
$$g(x) = \int_0^x f(s)ds - mx$$
is 2π-periodic.

Solution:

a.) Since $f(x) = f(x+2\pi)$ it follows that $f'(x+2\pi) = f'(x)$.

b.)
$$g(x+2\pi) = \int_0^{x+2\pi} f(s)ds$$
$$= \int_0^{x} f(s)ds + \int_x^{x+2\pi} f(s)ds$$
$$= g(x) + \int_0^{2\pi} f(u+x)du$$
$$= g(x).$$

C.) $g(x+2\pi) = \int_{x}^{x+2\pi} f(s)\,ds - m(x+2\pi)$

$$= \int_{0}^{x} f(s)\,ds + \int_{0}^{2\pi} f(u+x)\,du - mx - 2\pi m$$

$$= \int_{0}^{x} f(s)\,ds + 2\pi m - mx - 2\pi m$$

$$= g(x).$$