MST 352/652
Homework #6

Due Date: February 27, 2019

1 Problems for everyone

1. pg. 81-82, #3.2.14, #3.2.15, #3.2.17, #3.2.20.
2. pg. 87-88, #3.2.31-#3.2.35, #3.2.40.
3. pg. 95, #3.3.1-3.3.4.
4. pg. 97, #3.4.3.
5. pg. 138-140, #4.1.7-4.1.9, #4.1.12-4.1.15

2 Graduate Problems

1. Find the Fourier series of the function $|\sin(x)|$ in the interval $(-\pi, \pi)$. Use it to find the following sums:
$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$
2. Let $\phi(x) = x$.
   (a) Find the Fourier series of $\phi(x)$ on $(0, l)$.
   (b) Integrating term by term, find the Fourier series of $x^2/2$ on $(0, l)$. Be careful about calculating the a_0 term.
   (c) Find the sum of the following series:
   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$
   (d) Find the Fourier series of x^3 and x^4 on the interval $(0, l)$.
   (e) Find the sum of the following series:
   $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$
3. pg. 139, #4.1.16-4.1.18.

Homework #6

#3.2.15

Graph the following continuous functions. List all discontinuities and jump magnitudes.

a.) $f(x) = \begin{cases} e^x, & 1<|x|<2 \\ 0, & o.w \end{cases}$

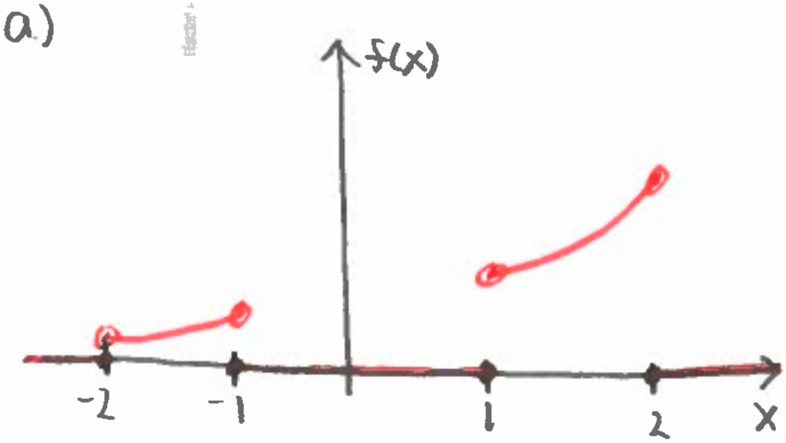
c.) $f(x) = \begin{cases} \frac{\sin(x)}{x} & 0<|x|<2\pi \\ 1 & x=0 \\ 0 & o.w \end{cases}$

d.) $f(x) = \begin{cases} x, & |x| \le 1 \\ x^2, & |x|>1 \end{cases}$

f.) $f(x) = \begin{cases} -1/x, & |x| \ge 1 \\ \frac{2}{1+x^2}, & |x|<1 \end{cases}$

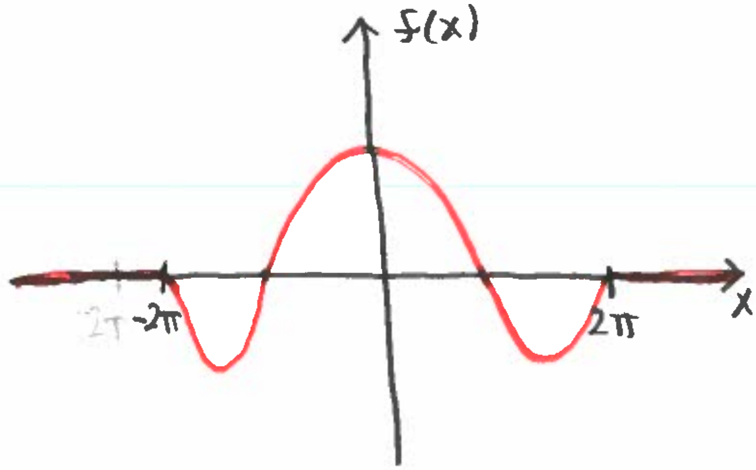
Solution:

a)



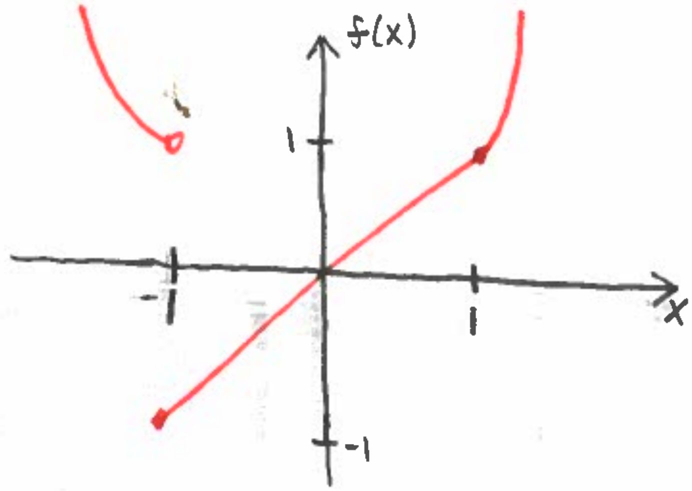
There are jump discontinuities at $x=-2,-1,1,2$ of magnitude $e^{-2}, e^{-1}, e^{1}, e^{2}$, respectively.

c.)



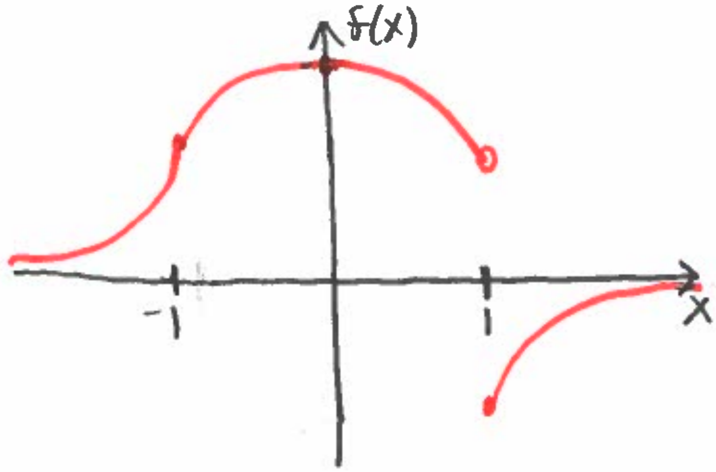
This function is continuous.

d.)



There is a jump discontinuity at x=-1 of magnitude 2.

f.)



There is a jump discontinuity at x=1 of magnitude 2.

#3.2.40

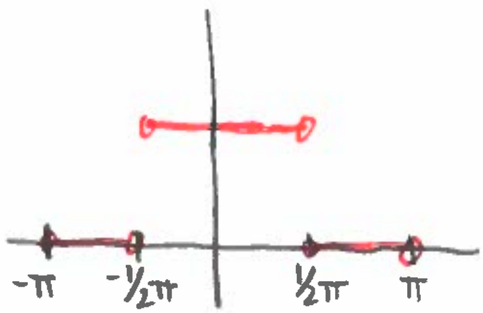
Find the Fourier series and discuss convergence for

a.) $b(x) = \begin{cases} 1, & |x| < \frac{1}{2}\pi \\ 0 & \frac{1}{2}\pi < |x| < \pi \end{cases}$

b.) $h(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0 & 1 < |x| < \pi. \end{cases}$

Solution:

a.)



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} b(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} b(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(nx)\,dx$$

$$= \frac{1}{n\pi}\sin(nx)\Big|_{-\pi/2}^{\pi/2}$$

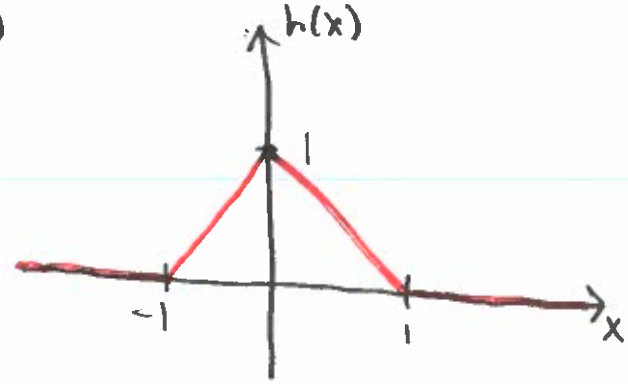
$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow b(x) \sim \frac{1}{2} + \frac{2}{\pi}\left(\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + \dots\right)$$

$$\sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\cos((2n-1)x)}{2n-1}.$$

This series will converge in L^2 but not pointwise to $b(x)$.

b.)



$$a_0 = \frac{2}{\pi}\int_0^1 (1-x)\,dx$$
$$= \frac{2}{\pi}\left(x - \frac{x^2}{2}\right)\Big|_0^1$$
$$= -\frac{1}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^1 (1-x)\cos(nx)\,dx$$
$$= \frac{2}{\pi n^2}(1-\cos(n))$$
$$\Rightarrow h(x) \sim -\frac{1}{2\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1-\cos(n))}{n^2}\cos(nx).$$

This Fourier series converges uniformly.

#3.3.1

Use integration to:

a) Find the Fourier series for $g(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}$

b.) Find the Fourier series for $g_2(x) = \begin{cases} \frac{1}{2}x^2, & x > 0 \\ 0, & x < 0 \end{cases}$

Solution:

Let

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$$\Rightarrow f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}$$

$$\Rightarrow g(x) = C + \frac{1}{2}x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

$$\Rightarrow g(x) = \frac{a_0}{2} + \sum^{\infty} \frac{(-1)^{n+1}}{(2n-1)}\sin((2n-1)x) - \frac{2}{\ }\sum^{\infty} \frac{\cos((2n-1)x)}{\ }.$$

Calculating, we have that

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx$$
$$= \frac{\pi}{2}$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin((2n-1)x) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}$$

b.) Integrating it follows that

$$f_2(x) = c + \frac{\pi}{4}x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2}\cos((2n-1)x) - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)^3}$$

$$= \frac{a_0}{2} + \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \sin((2n-1)x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2}\cos((2n-1)x)$$
$$- \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{(2n-1)^3}.$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}x^2\,dx$$
$$= \frac{\pi^2}{6}$$

$$\Rightarrow f_2(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\left(\frac{\pi}{2}\frac{(-1)^{n+1}}{(2n-1)} - \frac{2}{\pi}\cdot\frac{1}{(2n-1)^3}\right)\sin((2n-1)x) + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\cos((2n-1)x).$$

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#4.1.9

Solve the heat equation when the right hand side is kept fixed at a constant temperature α while the left hand side is insulated.

Solution:

If the rod has length L then we are interested in solving the following:

$$u_t = \gamma u_{xx}$$
$$u_x(t,0) = 0$$
$$u(t,L) = \alpha$$
$$u(0,x) = f(x).$$

We first find the steady state solution $u^*(x)$:

$$u^*_{xx} = 0$$
$$\Rightarrow u^*(x) = Ax + B$$

The boundary conditions imply $A=0$, $B=\alpha$.

Let $v(t,x) = u(t,x) - u^*(x)$. It follows that

$$V_t = \gamma V_{xx}$$
$$V_x(t,0) = 0$$
$$V(t,L) = 0$$
$$V(0,x) = f(x) - \alpha$$

Seperating variables it follows that:

$$X'' = -w^2 X, \quad T' = -w^2 T$$
$$\Rightarrow X = A\cos(wx) + B\sin(wx)$$
$$\Rightarrow X'(0) = 0 \Rightarrow B = 0$$
$$X(L) = 0 \Rightarrow w = \frac{(2n-1)\pi}{2L}.$$

Therefore,

$$V(t,x) = \sum_{n=1}^{\infty} a_n e^{-\frac{(2n-1)^2\pi^2}{4L^2}t} \cos\left(\frac{(2n-1)\pi}{2L}x\right)$$
$$a_n = \frac{2}{L}\int_0^L (f(x) - \alpha)\cos\left(\frac{(2n-1)\pi}{2L}x\right)dx.$$

#4.1.12

Show that the time derivative $v = u_t$ of any solution to the heat equation is also a solution. If $u(0,x) = f(x)$, what initial condition does $v(t,x)$ inherit?

Solution:

Since $V = U_t$ it follows that $V = U_{xx}$. Therefore,

$$\begin{aligned} V_t &= U_{tt} \\ &= (U_{xx})_t \\ &= U_{txx} \\ &= (U_t)_{xx} \\ &= U_{xxxx} \\ &= V_{xx}. \end{aligned}$$

Now, $V(0,x) = U_t(0,x) = U_{xx}(0,x) = f''(x)$.

#4.1.7

A metal bar of length $\ell=1$ and thermal diffusivity $\gamma=1$ is fully insulated. Suppose

$$u(0,x)=\begin{cases} x, & 0\le x\le \frac{1}{2} \\ 1-x, & \frac{1}{2}\le x\le 1 \end{cases}$$

a.) Use Fourier series to write down the temperature distribution at $t>0$.

b.) What is the equilibrium distribution?

c.) How fast does the solution go to equilibrium?

d.) Just before equilibrium what does the solution look like?

Solution:

a.) We want to solve:

$$u_t=u_{xx}$$
$$u_x(t,0)=0$$
$$u_x(t,1)=0$$
$$u(0,x)=\begin{cases} x, & 0\le x\le \frac{1}{2} \\ 1-x, & \frac{1}{2}\le x\le 1 \end{cases}$$

Seperable solutions satisfy:

$$X''=\lambda X,\quad T'=\lambda T$$

$\lambda=0$:

$$X=\frac{a_0}{2}$$

$\lambda<0$:

$$X=A\cos(\omega x)+B\sin(\omega x)$$
$$X'(0)=0 \Rightarrow B=0.$$
$$X'(1)=0 \Rightarrow \omega=n\pi.$$

Therefore, the generic form of the solutions is given by:

$$u(t,x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\cos(n\pi x).$$

$$\Rightarrow u(0,x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(n\pi x).$$

Consequently,

$$\int_0^1 u(0,x)\,dx = \int_0^1 \frac{a_0}{2}\,dx$$

$$\Rightarrow a_0 = 2\int_0^1 u(0,x)\,dx$$
$$= 2\cdot\frac{1}{4}$$
$$= \frac{1}{2}.$$

Also,

$$\int_0^1 u(0,x)\cos(n\pi x)\,dx = \int_0^1 a_n\cos^2(n\pi x)\,dx$$

$$\Rightarrow a_n = 2\int_0^{1/2} x\cos(n\pi x)\,dx + 2\int_{1/2}^1 (1-x)\cos(n\pi x)\,dx$$

$$\Rightarrow a_n = \frac{8\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)}{n^2\pi^2}$$

$$\Rightarrow u(t,x) = \frac{1}{4} + \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)}{n^2} e^{-n^2\pi^2 t}\cos(n\pi x)$$

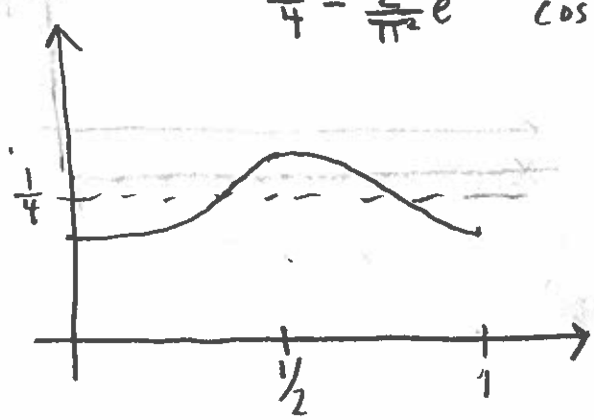
b.) The equilibrium distribution is given by:

$$\lim_{t\to\infty} u(t,x) = 1/4.$$

c.) The "rate" is $e^{-4\pi^2 t}$, i.e. the first decaying Fourier mode.

d.) The lowest order approximation is given by:

$$u(t,x) \simeq \frac{1}{4} + \frac{8}{\pi^2}\frac{(-1)}{4}e^{-4\pi^2 t}\cos(2\pi x)$$
$$= \frac{1}{4} - \frac{2}{\pi^2}e^{-4\pi^2 t}\cos(2\pi x)$$



Graduate Problems

#2.

Let $\phi(x) = x$.

a.) Find the Fourier series of $\phi(x)$ on $(0,\ell)$.

b.) Find the Fourier series of $x^2/2$ on $(0,\ell)$

c.) Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

d.) Find the Fourier series of x^3 and x^4 on $(0,\ell)$.

e.) Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

Solution:

a.) We can calculate the Fourier series on $(-\ell,\ell)$ and just truncate to $(0,\ell)$.

$$\begin{aligned} b_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} x\sin\left(\frac{n\pi x}{\ell}\right)dx \\ &= \frac{2}{\ell}\int_0^{\ell} x\sin\left(\frac{n\pi x}{\ell}\right)dx \\ &= \frac{2}{\ell}\cdot\frac{\ell}{n\pi}\left(-x\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell} + \int_0^{\ell}\cos\left(\frac{n\pi x}{\ell}\right)dx\right) \\ &= \frac{2\ell}{n\pi}(-1)^{n+1} \end{aligned}$$

$$\Rightarrow \phi(x) \sim \frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{\ell}\right)$$

b.) $$\frac{x^2}{2} \sim \frac{a_0}{2} + \frac{2\ell}{\pi}\cdot\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{\ell}\right)$$

$$\begin{aligned} a_0 &= \frac{1}{\ell}\int_{-\ell}^{\ell}\frac{x^2}{2}dx \\ &= \frac{1}{\ell}\int_0^{\ell} x^2dx \\ &= \frac{\ell^2}{3}. \end{aligned}$$

$$\Rightarrow \frac{x^2}{2} \sim \frac{\ell^2}{6} + \frac{2\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{\ell}\right).$$

c.) Therefore,

$$0 = \frac{\ell^2}{6} + \frac{2\ell^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

d.) $$\frac{x^3}{6} \sim \frac{a_0}{2} + \frac{\ell^2}{6}x + \frac{2\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$= \frac{\ell^2}{6}\cdot\frac{2\ell}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{\ell}\right) + \frac{2\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow x^3 \sim \frac{2\ell^3}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{\ell}\right) + \frac{12\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{\ell}\right)$$

$$\Rightarrow \frac{x^4}{4} \sim \frac{2\ell^4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{\ell}\right) + \frac{12\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos\left(\frac{n\pi x}{\ell}\right) + \frac{a_0}{2}$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell}\frac{x^4}{4}dx$$

$$= \frac{1}{2\ell}\int_0^{\ell}x^4dx$$

$$= \frac{\ell^4}{10}$$

$$\Rightarrow x^4 \sim \frac{\ell^4}{5} + \frac{8\ell^4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{\ell}\right) + \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos\left(\frac{n\pi x}{\ell}\right)$$

e.) Therefore,

$$0 = \frac{\ell^4}{5} + \frac{8\ell^4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + \frac{48\ell^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

$$\Rightarrow -\frac{1}{5} + \frac{2\pi^2}{3\pi^2} = \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

$$\Rightarrow \frac{7}{15} = \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^4} = -\frac{7\pi^4}{720}$$

#4.1.18

Solve the lossy convection-diffusion equation

$$u_t = \gamma u_{xx} + c u_x - \alpha u.$$

Solution:

Let $v(t,x) = u(t, x - ct)$. Then

$$v_t = u_t - c u_x$$
$$v_{xx} = u_{xx}$$

$$\Rightarrow v_t + c u_x = \gamma v_{xx} + c u_x - \alpha v$$
$$\Rightarrow v_t = \gamma v_{xx} - \alpha v.$$

Let $v = e^{-\alpha t} w$. Therefore,

$$v_t = -\alpha e^{-\alpha t} w + e^{-\alpha t} w_t$$
$$v_{xx} = e^{-\alpha t} w_{xx}$$

$$\Rightarrow w_t = \gamma w_{xx}.$$

w can be found by solving the heat equation. The solution is given by:

$$u(t,x) = v(t, x+ct)$$
$$= e^{-\alpha t} w(t, x+ct).$$

#1.

Find the Fourier series of $f(x) = |\sin(x)|$ on the interval $(-\pi, \pi)$. Use it to find the following sums:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}.$$

Solution:

Since $f(x)$ is even it follows that:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\,dx$$
$$= \frac{2}{\pi}\left(-\cos(x)\Big|_0^{\pi}\right)$$
$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$
$$= \frac{2(1+\cos(n\pi))}{\pi(1-n^2)}$$

Therefore,

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nx)}{1-4n^2}.$$

Consequently,

1. $f(0) = 0 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-4n^2}$

   $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$

2. $f(\pi/2) = 1 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2}.$

   $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}.$