MST 352/652
Homework #7

Due Date: March 5, 2019

1 Problems for everyone

1. pg. 185-186, #5.1.1-5.1.3, #5.1.5-5.1.7.

Homework #7

#5.1.2

Use the centered difference formula with step sizes $h = .1, .01, .001$, to approximate the derivative $u'(1)$ of the following functions

a.) x^4

b.) $\frac{1}{1+x^2}$

d.) $\cos(x)$

e.) $\tan^{-1}(x)$.

Solution:

a.) Let $f(x) = x^4$. Then,

$$f'(1) = \frac{(x+h)^4 - (x-h)^4}{2h} + O(h^2)$$

Therefore, for $h = .1, .01, .001$, the approximations are given by:

$$f'(1) \approx 4.04$$
$$\approx 4.0004$$
$$\approx 4.000004$$

We can see that the error grows like h^2.

b.) Let $f(x) = \frac{1}{1+x^2}$. Then,

$$f'(1) = \left(\frac{1}{1+(x+h)^2} - \frac{1}{1+(x-h)^2}\right)/2h + O(h^2).$$

$$\Rightarrow f'(1) \approx -.4999875$$
$$\approx -.49999999875$$
$$\approx -.499999999999875$$

d.) Let $f(x) = \cos(x)$. Then,

$$f'(1) = \frac{\cos(1+h) - \cos(1-h)}{2h} + O(h^2).$$

$$\Rightarrow f'(1) \approx -.840069$$
$$\approx -.841457$$
$$\approx -.841471$$

e.) Let $f(x) = \tan^{-1}(x)$. Then,

$$f'(1) = \frac{\tan^{-1}(1+h) - \tan^{-1}(1-h)}{2h}$$

$$\Rightarrow f'(1) \approx .500831$$
$$\approx .500008$$
$$\approx .500000.$$

#5.1.3

Approximate the second derivative $u''(1)$ of the functions in 5.1.2. using centered differences with $h=.1, .01, .001$.

Solution:

a.) Let $f(x)=x^4$. Then,

$$f''(1)=\frac{(x+h)^4-2x^4+(x-h)^4}{h^2}+O(h^2)$$

$$\Rightarrow f''(1)\approx 12.02$$
$$\approx 12.0003$$
$$\approx 12.000002$$

b.) Let $f(x)=\frac{1}{1+x^2}$. Then

$$f''(1)=\left(\frac{1}{1+(1+h)^2}-\frac{2}{1+1}+\frac{1}{1+(1-h)^2}\right)/h^2+O(h^2)$$

$$\Rightarrow f''(1)\approx .497$$
$$\approx .49997$$
$$\approx .4999997$$

d.) Let $f(x)=\cos(x)$. Then,

$$f''(1)=\frac{\cos(1+h)-2\cos(1)+\cos(1-h)}{h^2}+O(h^2)$$

$$\Rightarrow f''(1)\approx -.53985$$
$$\approx -.540298$$
$$\approx -.540302$$

e.) Let $f(x)=\tan^{-1}(x)$. Then,

$$f''(1)=\frac{\tan^{-1}(1+h)-2\tan^{-1}(1)+\tan^{-1}(1-h)}{h^2}+O(h^2).$$

$$\Rightarrow f''(1)\approx -.499996\ldots$$
$$\approx -.500000\ldots$$
$$\approx -.500000\ldots$$

$\blacksquare$

#5.1.5

Find a finite difference formula for $u''(x)$ using the points $x, x+h, x+2h$. What is the order of your approximation. Test your formula by computing the second derivative of $u(x)=e^{x^2}$ at $x=1$ and $x=0$.

Solution:

Taylor expanding it follows that:

$$u(x+h)=u(x)+u'(x)h+\tfrac{1}{2}u''(x)h^2+\tfrac{1}{6}u'''(x)h^3+\cdots$$
$$u(x+2h)=u(x)+2u'(x)h+2u''(x)h^2+\tfrac{8}{6}u'''(x)h^3+\cdots$$

$$\Rightarrow 2u(x+h)-u(x+2h)=u(x)-u''(x)h^2+O(h^3)$$
$$\Rightarrow u''(x)=\frac{u(x)-2u(x+h)+u(x+2h)}{h^2}+O(h)$$

Therefore, if $u(x)=e^{x^2}$ it follows that

$$u''(x)=\frac{e^{x^2}-2e^{(x+h)^2}+e^{(x+2h)^2}}{h^2}+O(h)$$

$\Rightarrow u''(1)\approx 23.2008$, $u''(0)\approx 2.07104$
≈ 16.8656 , $u''(0)\approx 2.00070$
≈ 16.3642 , $u''(0)\approx 2.00001$

#5.1.6

Using the function values $u(x)$, $u(x+h)$, $u(x+3h)$, construct a numerical approximation to the derivative $u'(x)$.

Solution:

Taylor expanding it follows that:

$$u(x+h)=u(x)+u'(x)h+\tfrac{1}{2}u''(x)h^2+\tfrac{1}{6}u'''(x)h^3+\cdots$$
$$u(x+3h)=u(x)+3u'(x)h+\tfrac{9}{2}u''(x)h^2+\tfrac{27}{6}u'''(x)h^3+\cdots$$

$$\Rightarrow 9u(x+h)-u(x+3h)=8u(x)+6u'(x)h+O(h^3)$$
$$\Rightarrow u'(x)=\frac{-8u(x)+9u(x+h)-u(x+3h)}{6h}+O(h^2).$$

#5.1.7

Answer #5.1.6, for the second derivative $u''(x)$.

Solution:

Taylor expanding it follows that

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + \dots$$
$$u(x+3h) = u(x) + 3u'(x)h + \tfrac{9}{2}u''(x)h^2 + \tfrac{27}{6}u'''(x)h^3 + \dots$$

$$\Rightarrow 3u(x+h) - u(x+3h) = 2u(x) - 3u''(x)h^2 + O(h^3)$$
$$\Rightarrow u''(x) = \frac{2u(x) - 3u(x+h) + u(x+3h)}{3h^2} + O(h).$$