MST 352/652
Homework #8

Due Date: March 19, 2019

1 Problems for everyone

1. pg. 149-151, #4.2.3, #4.2.4, #4.2.7, #4.2.8
2. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = u_{xx},\ x \in [0, 2\pi], \\ &u(0,t) = 1,\ u(2\pi,t), = 2 \\ &u(x,0) = x. \end{aligned}$$

   (a) Calculate the steady state solution for this initial-boundary value problem.
   (b) Solve this initial boundary value problem.

3. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = u_{xx} - u,\ x \in [0, 1], \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = x(1-x). \end{aligned}$$

   (a) Using separation of variables solve this initial-boundary value problem.
   (b) Using your solution, calculate $\lim_{t\to\infty} u(x,t)$.
   (c) Are there any steady state solutions to this equation? If so, what are they?

4. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_{tt} = u_{xx},\ x \in [0, \pi], \\ &u_x(0,t) = u_x(\pi,t) = 0, \\ &u(x,0) = \cos^2(x), \\ &u_t(x,0) = \cos(3x). \end{aligned}$$

   (a) Solve this initial-boundary value problem.
   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.
   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)
   (d) Describe qualitatively the behavior of the solution.

2 Graduate Problems

1. pg. 149-151, #4.2.10, #4.2.11

2. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_{tt} + u_t = u_{xx},\ x \in [0, \pi], \\ &u(0,t) = u(\pi, t) = 0, \\ &u(x,0) = \sin^2(x), \\ &u_t(x,0) = 0. \end{aligned}$$

   (a) Solve this initial-boundary value problem.

   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$ using 20 terms in the Fourier series.

   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

   (d) Describe qualitatively the behavior of the solution.

Homework #8

#4.2.3.

Write down the solutions to the initial-boundary value problems for the wave equation in the form of a Fourier series.

b.) $U_{tt} = 2U_{xx}$
$u(t,0) = u(t,\pi) = 0$
$u(0,x) = 0$
$U_t(0,x) = 1$

c.) $U_{tt} = 3U_{xx}$
$U(t,0) = U(t,\pi) = 0$
$U(0,x) = \sin^3(x)$
$U_t(0,x) = 0$

f.) $U_{tt} = 2U_{xx}$
$U_x(t,0) = U_x(t,2\pi) = 0$
$U(0,x) = -1$
$U_t(0,x) = 1$

g.) $U_{tt} = U_{xx}$
$U_x(t,0) = U_x(t,1) = 0$
$U(0,x) = x(1-x)$
$U_t(0,x) = 0$.

Solution:

b.) Seperating variables it follows that

$$\frac{T''}{2T} = \frac{X''}{X} = -w^2$$

Boundary conditions imply

$$X = A\sin(nx)$$
$$T = B\sin(\sqrt{2}nt) + C\cos(\sqrt{2}nt)$$
$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} \left(a_n \sin(\sqrt{2}nt) + b_n \cos(\sqrt{2}nt)\right)\sin(nx).$$

We now apply the initial conditions:

$$U(0,x) = 0 = \sum_{n=1}^{\infty} b_n \sin(nx)$$
$$\Rightarrow b_n = 0.$$

$$U_t(0,x) = 1 = \sum_{n=1}^{\infty} \sqrt{2}\, n a_n \sin(nx)$$

$$\Rightarrow \int_0^{\pi} \sin(nx)\,dx = \sqrt{2}\, n a_n \int_0^{\pi} \sin^2(nx)\,dx$$

$$\Rightarrow \int_0^{\pi} \sin(nx)\,dx = \frac{\sqrt{2}\, n\pi}{2} a_n$$

$$\Rightarrow a_n = \frac{\sqrt{2}}{\pi n}\left(-\frac{1}{n}(\cos(n\pi)-1)\right)$$

$$= \frac{\sqrt{2}}{\pi n^2}\left((-1)^{n+1}+1\right)$$

$$\Rightarrow U(t,x) = \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin(\sqrt{2}(2n-1)t)\sin((2n-1)x).$$

c.) First, I will simplify $\sin^3(x)$. Expanding it follows that

$$\sin^3(x) = \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 = -\frac{1}{8i}\left(e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}\right)$$

$$= -\frac{1}{8i}\left(e^{3ix} - e^{-3ix} - 3(e^{ix} - e^{-ix})\right)$$

$$= -\frac{\sin(3x)}{4} + \frac{3\sin(x)}{4}.$$

Following part (a), the generic solution is of the form

$$u(t,x) = \sum_{n=1}^{\infty} a_n \cos(\sqrt{3}\, nt)\sin(nx)$$

$$\Rightarrow -\frac{\sin(3x)}{4} + \frac{3\sin(x)}{4} = \sum_{n=1}^{\infty} a_n \sin(nx)$$

$$\Rightarrow u(t,x) = -\frac{1}{4}\cos(3\sqrt{3}t)\sin(3x) + \frac{3}{4}\cos(\sqrt{3}t)\sin(x).$$

:.) Seperating variables it follows that

$$\frac{T''}{2T} = \frac{X''}{X} = -w^2$$

Boundary conditions imply the spatial eigenfunctions are

$$X = 1 \quad (n=0), \quad X = \cos\left(\frac{nx}{2}\right) \quad (n>0).$$

$$\Rightarrow u(t,x) = c + dt + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\sqrt{2}nt}{2}\right)\cos(nx) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\sqrt{2}nt}{2}\right)\cos\left(\frac{nx}{2}\right)$$

Initial conditions imply:

$$-1 = c + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{and} \quad 1 = d + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{2} n b_n \cos\left(\frac{nx}{2}\right)$$

$$\Rightarrow c = -1,\ a_n = 0 \quad \text{and} \quad 1 = d,\ b_n = 0$$

$$\Rightarrow u(t,x) = -1 + t.$$

9.) Seperating variables it follows that

$$\frac{T''}{T} = \frac{X''}{X} = -w^2$$

Boundary conditions imply that the spatial eigenvectors are

$$X = 1 \quad (n=0) \,, \quad X = \cos(n\pi x) \quad (n>0).$$

Therefore,

$$u(t,x) = c + dt + \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \cos(n\pi x)$$

Initial conditions imply

$$u(0,x) = x(1-x) = c + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$\Rightarrow \int_0^1 x(1-x)\,dx = \int_0^1 c\,dx$$

$$\Rightarrow c_n = \frac{1}{6}.$$

Also,

$$\int_0^1 x(1-x)\cos(n\pi x)\,dx = \frac{a_n}{2}$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}((-1)^{n+1}-1)$$

Therefore,

$$u(t,x) = \frac{1}{6} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi t)\cos((2n-1)\pi x)}{(2n-1)^2}.$$

#4.2.4

Find all seperable solutions to the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le \pi$ subject to

a.) Mixed boundary conditions

$$u(t,0) = u_x(t,\pi) = 0.$$

b.) Neumann boundary conditions

$$u_x(t,0) = u_x(t,\pi) = 0.$$

Solutions:

a.) $u_n(t,x) = \left(a_n \cos\left(\frac{(2n-1)\pi}{2} t\right) + b_n \sin\left(\frac{(2n-1)\pi}{2} t\right)\right) \sin\left(\frac{(2n-1)\pi}{2} x\right) \quad (n>0)$

b.) $u_n(t,x) = A + Bt \quad (n=0)$

$u_n(t,x) = (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \cos(n\pi x) \quad (n>0)$

#4.2.7

Show that if u satisfies the wave equation then $v = u_t$ also satisfies the wave equation. What initial conditions does v satisfy.

Solution:

Suppose

$$u_{tt} = u_{xx}$$
$$u(0,x) = f(x)$$
$$u_t(0,x) = g(x).$$

Let $v(t,x) = u_t(t,x)$. Therefore,

$$v_{tt} = u_{ttt} = u_{xxt} = v_{xx}$$

Moreover,

$$v(t,0) = u_t(t,0) = g(x)$$
$$v_t(t,0) = u_{tt}(t,0) = u_{xx}(t,0) = f''(x).$$

#2

Consider the following initial value problem:

$$u_t = u_{xx}, \quad x \in [0, 2\pi]$$
$$u(t,0) = 1$$
$$u(t,2\pi) = 2$$
$$u(0,x) = x.$$

a.) Calculate the steady state solution for this initial-boundary value problem.

b.) Solve this initial boundary value problem.

olution:

a.) The state state solution is $x+1$.

b.) Let $v = u - (x+1)$. Then, v satisfies the following intial-boundary value problem:

$$v_t = v_{xx}$$
$$v(t,0) = 0$$
$$v(t,2\pi) = 0$$
$$v(0,x) = -1.$$

Seperating variables it follows that

$$V(t,x)=\sum_{n=1}^{\infty} a_n e^{-\frac{n^2 t}{4}} \sin\left(\frac{nx}{2}\right)$$

$$\Rightarrow -1=\sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right)$$

$$\Rightarrow -\int_0^{2\pi} \sin\left(\frac{nx}{2}\right)dx=\pi a_n$$

$$\Rightarrow \frac{2}{\pi n}\cos\left(\frac{nx}{2}\right)\Big|_0^{2\pi}=a_n$$

$$\Rightarrow a_n=\frac{2}{\pi n}\left((-1)^n-1\right)$$

$$\Rightarrow U(t,x)=V(t,x)+(x+1)$$
$$=1+x+\sum_{n=1}^{\infty} a_n e^{-\frac{(2n-1)^2 t}{4}} \sin\left(\frac{(2n-1)x}{2}\right)$$

[redacted]

#4.

Consider the initial value problem

$$U_{tt}=U_{xx}$$
$$U_x(0,t)=U_x(\pi,t)=0$$
$$U(x,0)=\cos^2(x)$$
$$U_t(x,0)=\cos(3x)$$

a.) Solve this initial value problem

b.) Sketch the solutions for $t=0, \pi/2, \pi, 3\pi/2, 2\pi$.

c.) Sketch a contour plot of your solution as a function of x and t

Solution:

a.) The solution has the generic form:

$$U(t,x)=a+bt+\sum_{n=1}^{\infty} a_n \sin(nt)\cos(nx)+\sum_{n=1}^{\infty} b_n \cos(nt)\cos(nx).$$

$$U(0,x)=\frac{1}{2}+\frac{\cos(2x)}{2}$$

$$\Rightarrow a=1/2,\ b_2=1/2,\ b_n=0\ (n\neq 2)$$

$$U_t(0,x)=\cos(3x)=\sum_{n=1}^{\infty} n a_n \cos(nx)+b$$

$$\Rightarrow b=0,\ a_3=1/3,\ a_n=0\ (n\neq 3).$$

$$\Rightarrow u(t,x)=\frac{1}{2}+\frac{\cos(2t)\cos(2x)}{2}+\frac{\sin(3t)\cos(3x)}{3}.$$

b.) At the end

c.) At the end

Graduate Problems

#4.2.11

The initial boundary value problem

$$u_{tt}=-u_{xxxx}$$
$$u(t,0)=u(t,1)=0$$
$$u_{xx}(t,0)=u_{xx}(t,1)=0$$
$$u(0,x)=f(x)$$
$$u_t(0,x)=0$$

models the vibrations of a beam.

a.) What are the vibrational modes of the beam?

b.) Solve this equation.

c.) Does the beam vibrate periodically.

Solution:

a.) Seperating variables it follows that:

$$T''X=-TX''''$$

$$\Rightarrow -\frac{T''}{T}=\frac{X''''}{X}=\lambda$$

$$\Rightarrow X''''=\lambda X,\ T''=-\lambda T$$

The only case in which the boundary conditions can be satisfied is $\lambda>0$. Consequently,

$$u_n(t,x)=(a_n\sin(n^2\pi^2 t)+b_n\cos(n^2\pi^2 t))\sin(n\pi x).$$

Therefore, the vibrational frequencies are

$$\omega=\frac{1}{n^2\pi^2}$$

$$\Rightarrow u(t,x) = \sum_{n=1}^{\infty} \frac{8}{\pi(2n-1)(4-(2n-1)^2)} e^{-t/2} \cos\left(\frac{\sqrt{(2n-1)^2-1}}{2} t\right) \sin\left((2n-1)\right.$$

b.) At the end.

c.) At the end.

b.) The Fourier coefficients are given by

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

$$a_n = 0.$$

c.) Yes.

#2.

Consider the following initial-boundary value problem:

$$U_{tt} + U_t = U_{xx}$$

$$U(0,t) = U(\pi,t) = 0$$

$$U(x,0) = \sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$$

$$U_t(x,0) = 0$$

a.) Solve this initial value problem.

b.) Sketch the solution for $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$

c.) Sketch a contour plot of your solution.

Solution:

a.) Seperating variables it follows that

$$T''X + T'X = TX''$$

$$\Rightarrow T'' + T' = \lambda T, \quad X'' = \lambda X.$$

Assuming $\lambda = -\omega^2 < 0$ it follows that the family of solutions are given by:

$$U_n(t,x) = \exp\left(-\frac{t}{2}\right)\left(a_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + b_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right)\sin(nx)$$

By the principle of linear super

$$U(t,x) = \exp\left(-\frac{t}{2}\right)\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + b_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right)\sin(nx)$$

$$\Rightarrow \frac{1}{2} - \frac{\cos(2x)}{2} = \sum_{n=1}^{\infty} a_n\sin(nx)$$

$$\Rightarrow \frac{2((-1)^n-1)}{n(n^2-4)} = \frac{\pi a_n}{2}$$

$$\Rightarrow a_n = \frac{4((-1)^n-1)}{\pi n(n^2-4)}$$

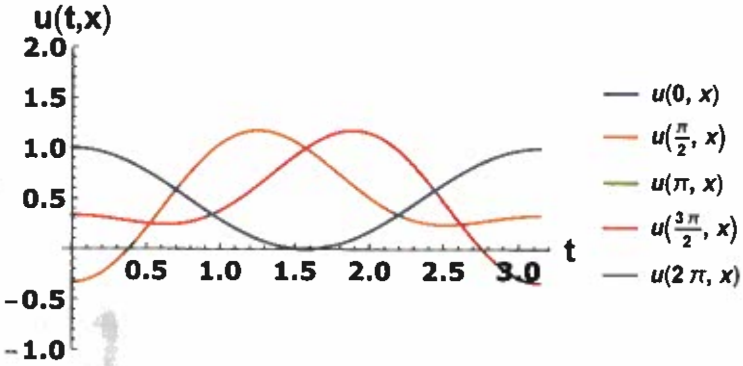
#4

In[201]:=
```
u[t_, x_] := 1 / 2 + Cos[2 * t] * Cos[2 * x] / 2 + Sin[3 * t] * Cos[3 * x] / 3
```

Snapshots in time

In[209]:=
```
Plot[{u[0, x], u[π / 2, x], u[π, x], u[3 * π / 2, x], u[2 * π, x]}, {x, 0, π},
  PlotRange → {-1, 2}, AxesLabel → {"t", "u(t,x)"}, PlotLegends → "Expressions",
  AxesStyle → {Black, Black}, TicksStyle → Directive["Label", 14]]
```

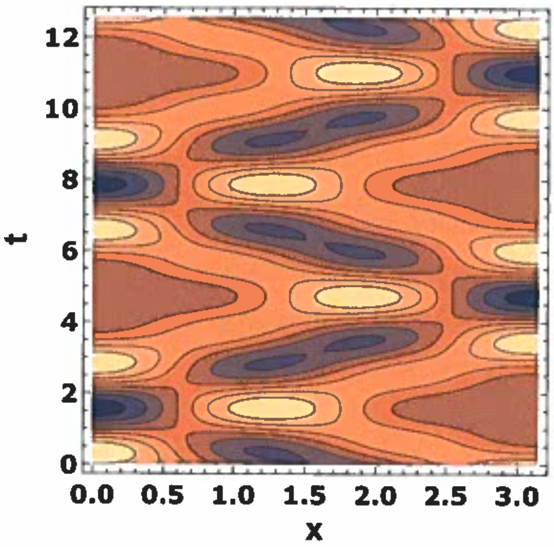
Out[209]=



Contour Plot

In[219]:=
```
ContourPlot[u[t, x], {x, 0, π}, {t, 0, 4 * π},
  FrameLabel → {"x", "t"}, PlotLegends → "Expressions",
  FrameStyle → {Black, Black}, FrameTicksStyle → Directive["Label", 14]]
```

Out[219]=



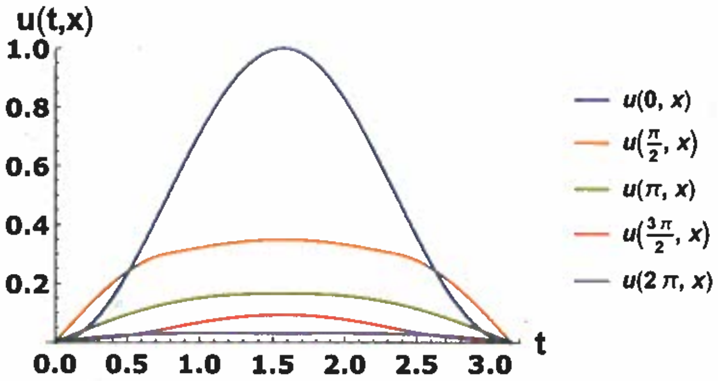
GP #2

```
In[222]:= u[t_, x_] := Exp[-t / 2] * Sum[8 / (π * (2 * n - 1) * (4 - (2 * n - 1) ^ 2)) *
          Cos[Sqrt[(2 * n - 1) ^ 2 - 1] / 2 * t] * Sin[(2 * n - 1) * x], {n, 1, 20}]
```

Snapshots in time

```
In[224]:= Plot[{u[0, x], u[π / 2, x], u[π, x], u[3 * π / 2, x], u[2 * π, x]}, {x, 0, π},
          PlotRange → {0, 1}, AxesLabel → {"t", "u(t,x)"}, PlotLegends → "Expressions",
          AxesStyle → {Black, Black}, TicksStyle → Directive["Label", 14]]
```

Out[224]=



Contour Plot

```
In[227]:= ContourPlot[u[t, x], {x, 0, π}, {t, 0, 2 * π}, FrameLabel → {"x", "t"},
          PlotLegends → "Expressions", FrameStyle → {Black, Black},
          FrameTicksStyle → Directive["Label", 14], PlotRange → {0, 1}]
```

Out[227]=