MST 352/652
Homework #9

Due Date: March 26, 2019

1 Problems for everyone

1. pg. 154-155, #4.3.3-#4.3.9
2. pg. 159-160, #4.3.10, #4.3.11, #4.3.13, #4.3.17, #4.3.18.
3. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = u_{xx},\ x \in [0, 2\pi], \\ &u(0,t) = u(2\pi, t) = 0, \\ &u(x,0) = \sin(x) - \sin(3x) + \sin(5x). \end{aligned}$$

   (a) Solve this initial-boundary value problem.

   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$.

   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

4. Consider the following initial-boundary value problem:

$$\begin{aligned} u_t &= u_{xx},\ x \in [0, 2\pi], \\ u_x(0,t) &= u_x(2\pi, t) = 0, \\ u(x,0) &= (x - \pi)^2. \end{aligned}$$

   (a) Solve this initial-boundary value problem.

   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$ by using the first 20 terms in the Fourier series.

   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

5. Consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = u_{xx},\ x \in [0, 2\pi], \\ &u(0,t) = u(2\pi, t), \\ &u_x(0,t) = u_x(2\pi, t), \\ &u(x,0) = x^2. \end{aligned}$$

   (a) What do the boundary conditions model for this problem?

(b) Using separation of variables, solve this initial-value problem.

(c) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$ by using the first 20 terms in the Fourier series.

(d) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

6. Consider the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= u_{xx},\ x \in [0,\pi], \\ u(0,t) &= 0, \\ u_x(\pi,t) &= 0, \\ u(x,0) &= \sin^2(x), \\ u_t(x,0) &= \cos^2(x). \end{aligned}$$

(a) Solve this initial-boundary value problem.

(b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

(c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

(d) Describe qualitatively the behavior of the solution.

2 Graduate Problems

1. The following problem models the vibrations of a square drum:

$$\begin{aligned} u_{tt} &= \Delta u,\ x \in \Omega, \\ u|_{u\in\partial\Omega} &= 0, \\ u(x,0) &= f(x,y), \\ u_t(x,0) &= 0, \end{aligned}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$.

(a) Assuming a separable solution of the form $u(x,t) = X(x)Y(y)T(t)$, derive eigenvalue problems that must be satisfied by X, Y and T. Using the boundary conditions determine the eigenvalues for this problem.

(b) Using linear superposition write down the solution to this problem in terms of a *double Fourier series.*

(c) Using orthogonality determine a formula for the coefficients expressed as *double integrals.*

(d) For $a, b \in \mathbb{N}$, solve this initial boundary value problem if $f(x,y) = \sin(ax)\sin(by)$.

(e) Sketch contour plots of the solution at $t = 0, t = \pi/2, t = \pi, t = 3\pi/2$ for $a = 1, b = 1$ and $a = 2, b = 2$. (If you want to, you can use software to do this)

(f) For $a, b \in \mathbb{N}$, describe qualitatively the behavior of the drum as a function of time.