MST 352/652
Homework #9

Due Date: March 26, 2019

1 Problems for everyone

1. pg. 154-155, #4.3.3-#4.3.9
2. pg. 159-160, #4.3.10, #4.3.11, #4.3.13, #4.3.17, #4.3.18.
3. Consider the following initial-boundary value problem:

$$u_t = u_{xx},\ x \in [0, 2\pi],$$
$$u(0,t) = u(2\pi, t) = 0,$$
$$u(x,0) = \sin(x) - \sin(3x) + \sin(5x).$$

   (a) Solve this initial-boundary value problem.
   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$.
   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

4. Consider the following initial-boundary value problem:

$$u_t = u_{xx},\ x \in [0, 2\pi],$$
$$u_x(0,t) = u_x(2\pi, t) = 0,$$
$$u(x,0) = (x-\pi)^2.$$

   (a) Solve this initial-boundary value problem.
   (b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$ by using the first 20 terms in the Fourier series.
   (c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

5. Consider the following initial-boundary value problem:

$$u_t = u_{xx},\ x \in [0, 2\pi],$$
$$u(0,t) = u(2\pi, t),$$
$$u_x(0,t) = u_x(2\pi, t),$$
$$u(x,0) = x^2.$$

   (a) What do the boundary conditions model for this problem?

(b) Using separation of variables, solve this initial-value problem.

(c) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = .1$, $t = .25$, $t = .5$, and $t = 1$ by using the first 20 terms in the Fourier series.

(d) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

6. Consider the following initial-boundary value problem:

$$\begin{aligned} u_{tt} &= u_{xx},\ x \in [0,\pi], \\ u(0,t) &= 0, \\ u_x(\pi,t) &= 0, \\ u(x,0) &= \sin^2(x), \\ u_t(x,0) &= \cos^2(x). \end{aligned}$$

(a) Solve this initial-boundary value problem.

(b) Using software such as Matlab, Mathematica, etc sketch the solution for $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

(c) Sketch a contour plot of your solution as a function of x and t. (If you want to, you can use software to do this.)

(d) Describe qualitatively the behavior of the solution.

2 Graduate Problems

1. The following problem models the vibrations of a square drum:

$$\begin{aligned} u_{tt} &= \Delta u,\ x \in \Omega, \\ u|_{u\in\partial\Omega} &= 0, \\ u(x,0) &= f(x,y), \\ u_t(x,0) &= 0, \end{aligned}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$.

(a) Assuming a separable solution of the form $u(x,t) = X(x)Y(y)T(t)$, derive eigenvalue problems that must be satisfied by X, Y and T. Using the boundary conditions determine the eigenvalues for this problem.

(b) Using linear superposition write down the solution to this problem in terms of a *double Fourier series.*

(c) Using orthogonality determine a formula for the coefficients expressed as *double integrals.*

(d) For $a, b \in \mathbb{N}$, solve this initial boundary value problem if $f(x,y) = \sin(ax)\sin(by)$.

(e) Sketch contour plots of the solution at $t = 0, t = \pi/2, t = \pi, t = 3\pi/2$ for $a = 1, b = 1$ and $a = 2, b = 2$. (If you want to, you can use software to do this)

(f) For $a, b \in \mathbb{N}$, describe qualitatively the behavior of the drum as a function of time.

Homework #9

#4.3.8

Suppose $u(x,y)$ is a solution to Laplace's equation

a.) Show that $U(x,y) = u(x-a, y-b)$ is also a solution

b.) Show that $U(x,y) = u(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$ is also a solution.

Solution:

a) Let $w = x-a, v = y-b$. Then,

$$U_x = \frac{\partial w}{\partial x} U_w = U_w \quad \text{and} \quad U_y = \frac{\partial v}{\partial y} U_v = U_v$$

$$\Rightarrow U_{xx} + U_{yy} = U_{ww} + U_{vv} = 0.$$

b.) Let $w = x\cos\theta + y\sin\theta$, $v = -x\sin\theta + y\cos\theta$. Then,

$$U_x = \frac{\partial w}{\partial x} U_w + \frac{\partial v}{\partial x} U_v = \cos\theta U_w + \sin\theta U_v$$

$$U_y = \frac{\partial w}{\partial y} U_w + \frac{\partial v}{\partial y} U_v = -\sin\theta U_w + \cos\theta U_v$$

$$\Rightarrow U_{xx} = \cos^2\theta U_{ww} + 2\cos\theta\sin\theta U_{wv} + \sin^2\theta U_{vv}$$

$$U_{yy} = \sin^2\theta U_{ww} - 2\cos\theta\sin\theta U_{wv} + \cos^2\theta U_{ww}$$

$$\Rightarrow U_{xx} + U_{yy} = U_{ww} + U_{vv} = 0.$$

∎

#4.3.11

Use linear superposition to solve the following boundary value problem

$$\Delta u = 0$$
$$u(x,0) = f(x)$$
$$u(x,b) = g(x)$$
$$u(0,y) = h(y)$$
$$u(a,y) = k(y).$$

Solution:

We split this into four problems:

$\Delta V_1 = 0$	ΔV_2	$\Delta V_3 = 0$	$\Delta V_4 = 0$
$V_1(x,0)=f(x)$	$V_2(x,0)=0$	$V_3(x,0)=0$	$V_4(x,0)=0$
$V_1(x,b)=0$	$V_2(x,b)=g(x)$	$V_3(x,b)=0$	$V_4(x,b)=0$
$V_1(0,y)=0$	$V_2(0,y)=0$	$V_3(0,y)=h(y)$	$V_4(0,y)=0$
$V_1(a,y)=0$	$V_2(a,y)=0$	$V_4(a,y)=0$	$V_4(a,y)=K(y)$
PI	PII	PIII	PIV

PI:

$V_1 = X(x)Y(y)$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -w^2$$

$$\Rightarrow X_n = A_n \sin\left(\frac{n\pi}{a}x\right)$$

$$Y_n = B_n \cosh\left(\frac{n\pi}{a}y\right) + C_n \sinh\left(\frac{n\pi}{a}y\right)$$

$$Y_n(b) = 0 \Rightarrow B_n \cosh\left(\frac{n\pi b}{a}\right) + C_n \sinh\left(\frac{n\pi b}{a}\right)$$

$$\Rightarrow B_n = -C_n \tanh\left(\frac{n\pi b}{a}\right)$$

By linear superposition:

$$V_1 = \sum_{n=1}^{\infty} b_n\left(\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\right)\sin\left(\frac{n\pi}{a}x\right)$$

$$\Rightarrow V_1(x,0) = \sum_{n=1}^{\infty} -b_n \tanh\left(\frac{n\pi b}{a}\right)\sin(n\pi x) = f(x)$$

$$\Rightarrow b_n = -\frac{2}{a}\coth\left(\frac{n\pi b}{a}\right)\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

PII:

$V_2 = X(x)Y(y)$

$X_n = A_n \sin\left(\frac{n\pi}{a}x\right)$

$Y_n = B_n \sinh\left(\frac{n\pi}{a}y\right)$

By linear superposition:

$$V_2(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$b_n = \frac{2}{a\sinh\left(n\pi\frac{b}{a}\right)}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

PIII:

$V_3 = X(x)Y(y)$

$Y_n = A_n \sin\left(\frac{n\pi y}{b}\right)$

$X_n = B_n \left(\sinh\left(\frac{n\pi x}{b}\right) - \tanh\left(\frac{n\pi a}{b}\right)\cosh\left(\frac{n\pi x}{b}\right)\right)\sin\left(\frac{n\pi y}{b}\right)$

$\rightarrow V_3(x,y) = \sum_{n=1}^{\infty} b_n \left(\sinh\left(\frac{n\pi x}{b}\right) - \tanh\left(\frac{n\pi a}{b}\right)\cosh\left(\frac{n\pi x}{b}\right)\right)\sin\left(\frac{n\pi y}{b}\right)$

$b_n = -\frac{2}{b}\coth\left(\frac{n\pi a}{b}\right)\int_0^b h(y)\sin\left(\frac{n\pi}{b}y\right)dy$

PIV:

$V_4 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi x}{b}\right)$

$b_n = \frac{2}{a\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b K(y)\sin\left(\frac{n\pi}{b}y\right)dy$

The solution is then given by:

$u(x,y) = V_1(x,y) + V_2(x,y) + V_3(x,y) + V_4(x,y)$

■

#4.3.17

Use seperation of variables to solve

$u_{xx} + 2u_y + u_{yy} = 0$

$u(x,0) = 0$

$u(x,1) = f(x)$

$u(0,y) = 0$

$u(1,y) = 0.$

Solution:

Assume $u(x,y) = X(x)Y(y)$. Then,

$X''Y + 2XY' + XY'' = 0$

$\frac{X''}{X} = -\frac{2Y' - Y''}{Y} = -w^2$

Boundary conditions imply $w = n\pi$ and

$X(x) = A\sin(n\pi x)$

We also have:

$$Y''+2Y'-n^2\pi^2 Y=0$$

$$\Rightarrow Y=Be^{-y+\sqrt{1+n^2\pi^2}y}+Ce^{-y-\sqrt{1+n^2\pi^2}y}$$

$$Y(0)=0 \Rightarrow B=-C$$

$$\Rightarrow Y(y)=Be^{-y}\left(e^{\sqrt{1+n^2\pi^2}y}-e^{-\sqrt{1+n^2\pi^2}y}\right)$$

$$=2Be^{-y}\sinh\left(\sqrt{1+n^2\pi^2}y\right)$$

The generic solution is therefore

$$u(x,y)=e^{-y}\sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh\left(\sqrt{1+n^2\pi^2}y\right)$$

$$\Rightarrow f(x)=e^{-1}\sum_{n=1}^{\infty} b_n \sin(n\pi x)\sinh\left(\sqrt{1+n^2\pi^2}\right)$$

$$\Rightarrow b_n=\frac{2e}{\sinh\left(\sqrt{1+n^2\pi^2}\right)}\int_0^1 \sin(n\pi x)f(x)dx.$$

#3.

Consider the following initial-boundary value problem:

$$U_t=U_{xx}$$

$$U(t,0)=U(t,2\pi)=0$$

$$U(x,0)=\sin(x)-\sin(3x)+\sin(5x)$$

a.) Solve this initial value problem.

c.) Sketch a contour plot as a function of x and t.

Solution:

a.) Since $\sin(x), -\sin(3x), \sin(5x)$ are eigenfunctions it follows that

$$u(t,x)=e^{-t}\sin(x)-e^{-9t}\sin(3x)+e^{-25t}\sin(5x).$$

b.) (See Attached).

#5.

Consider the following initial value problem:

$$U_t = U_{xx}$$
$$U(t,0) = U(t,2\pi)$$
$$U_x(t,0) = U_x(t,2\pi)$$
$$U(x,0) = x^2$$

a.) What do the boundary conditions model?

b.) Solve this initial value problem.

c.) Sketch a contour plot.

Solution:

a.) Periodic boundary conditions, i.e heat flow in disk.

b.) The generic solution is

$$U(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) e^{-n^2 t}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = 8\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos(nx)\,dx = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin(nx)\,dx = -\frac{4\pi}{n}$$

c.) See Attached.

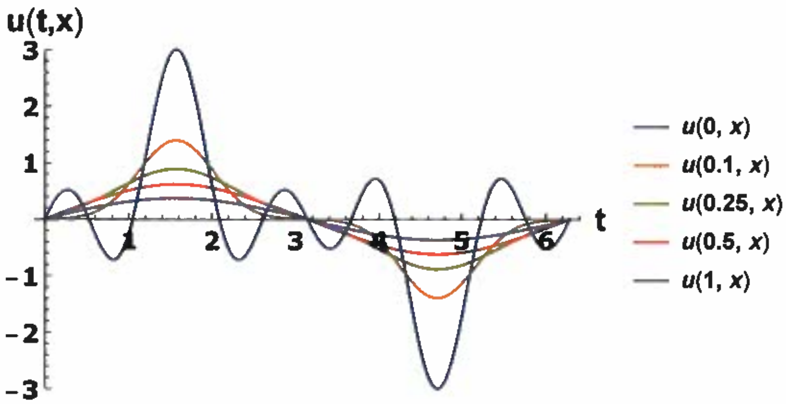
#3

In[288]:=
```
u[t_, x_] := Exp[-t] * Sin[x] - Exp[-9 * t] * Sin[3 * x] + Exp[-25 * t] * Sin[5 * x];
```

Snapshots in time

In[289]:=
```
Plot[{u[0, x], u[.1, x], u[.25, x], u[.5, x], u[1, x]}, {x, 0, 2 * π},
 PlotRange → {-3, 3}, AxesLabel → {"t", "u(t,x)"}, PlotLegends → "Expressions",
 AxesStyle → {Black, Black}, TicksStyle → Directive["Label", 14]]
```

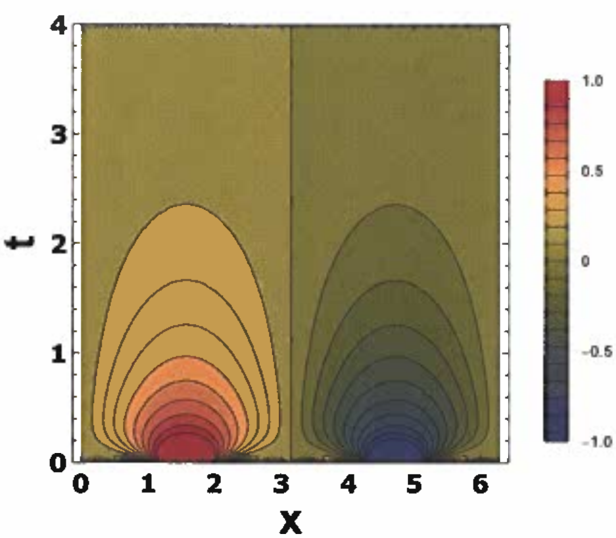
Out[289]=



Contour Plot

In[310]:=
```
ContourPlot[u[t, x], {x, 0, 2 * π}, {t, 0, 4}, FrameLabel → {"X", "t"},
 PlotLegends → Automatic, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1}, Contours → 20,
 PlotLegends → Automatic, ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
```

Out[310]=



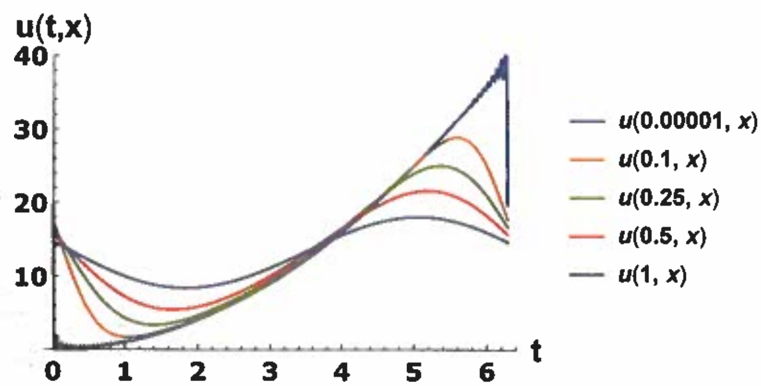
#3

In[326]:=
```
u[t_, x_] :=
  4 * π^2 / 3 + Sum[Exp[-n^2 * t] * (4 / n^2 * Cos[n * x] - 4 * π / n * Sin[n * x]), {n, 1, 100}];
```

Snapshots in time

In[327]:=
```
Plot[{u[.00001, x], u[.1, x], u[.25, x], u[.5, x], u[1, x]}, {x, 0, 2 * π},
 PlotRange → {0, 40}, AxesLabel → {"t", "u(t,x)"}, PlotLegends → "Expressions",
 AxesStyle → {Black, Black}, TicksStyle → Directive["Label", 14]]
```

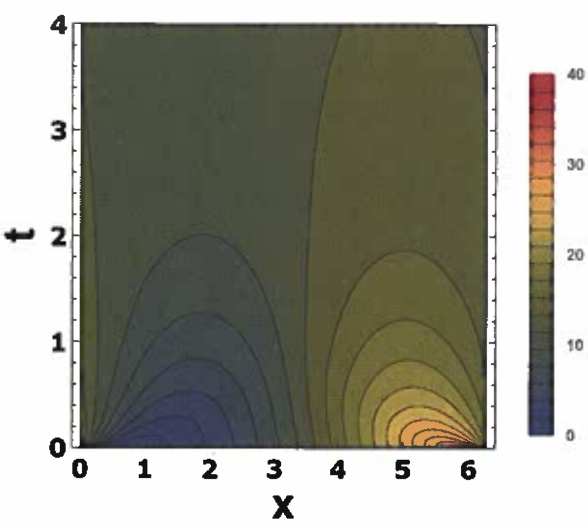
Out[327]=



Contour Plot

In[329]:=
```
ContourPlot[u[t, x], {x, 0, 2 * π}, {t, 0, 4}, FrameLabel → {"X", "t"},
 PlotLegends → Automatic, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {0, 40}, Contours → 20,
 PlotLegends → Automatic, ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
```

Out[329]=



Graduate Problem

#1

The following problem models the vibrations of a square drum

$$U_{tt} = \Delta U$$
$$U|_{\partial\Omega} = 0$$
$$U(0,x) = f(x,y)$$
$$U_t(0,x) = 0$$

Where $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$.

a.) Solve this problem

b.) For $a, b \in \mathbb{N}$, solve this boundary value problem if $f(x,y) = \sin(ax)\sin(by)$.

c.) Sketch contour plots of your solution at $t = 0, \pi/2, \pi, 3\pi/2$ for $a=1, b=1$ and $a=2, b=2$.

Solution:

a.) The generic form of the solution is

$$U(t,x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \cos(\sqrt{n^2+m^2}\,\pi t) \sin(n\pi x)\sin(n\pi y)$$

$$\Rightarrow f(x,y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_{nm} \sin(n\pi x)\sin(n\pi y)$$

$$\Rightarrow b_{nm} = 4 \int_0^1 \int_0^1 \sin(n\pi x)\sin(n\pi y) f(x,y)\,dx\,dy$$

b.) For $a=1, b=1$:

$$b_{nm} = 4\int_0^1 \int_0^1 \sin(n\pi x)\sin(n\pi y)\sin(x)\sin(y)\,dx\,dy$$
$$= \frac{4(-1)^{m+n} mn\pi^2 \sin^2(1)}{(-1+m^2\pi^2)(-1+n^2\pi^2)}$$

For $a=2, b=2$:

$$b_{nm} = \frac{4(-1)^{m+n} mn\pi^2 \sin^2(2)}{(-4+m^2\pi^2)(-4+n^2\pi^2)}$$

c.) See attached.

#Graduate Problem

a=1 and b=1

In[343]:=
```
Simplify[Integrate[Integrate[Sin[n * π * x] * Sin[m * π * y] * Sin[x] * Sin[y], {x, 0, 1}],
  {y, 0, 1}], Assumptions → n ∈ Integers && m ∈ Integers]
```

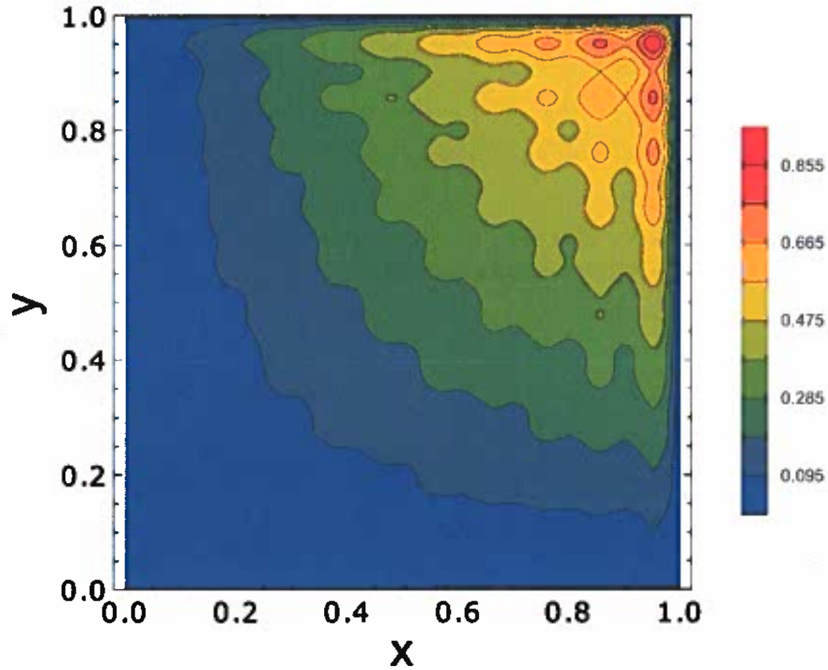
Out[343]=
$$\frac{(-1)^{m+n}\, m\, n\, \pi^2\, \mathrm{Sin}[1]^2}{\left(-1+m^2\pi^2\right)\left(-1+n^2\pi^2\right)}$$

In[362]:=
```
u[t_, x_, y_] :=
  4 * Sum[Sum[((-1)^(m+n) m n π^2 Sin[1]^2)/((-1 + m^2 π^2) (-1 + n^2 π^2)) * Cos[Sqrt[n^2 + m^2] * π * t] * Sin[n * π * x] *
      Sin[m * π * y], {n, 1, 20}], {m, 1, 20}];
```

Snapshots in time

In[373]:=
```
ContourPlot[u[0, x, y], {x, 0, 1}, {y, 0, 1},
 FrameLabel → {"x", "y"}, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 PlotLegends → Automatic, Contours → 20, ColorFunction → "DarkRainbow"]
ContourPlot[u[π / 2, x, y], {x, 0, 1}, {y, 0, 1},
 FrameLabel → {"x", "y"}, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 PlotLegends → Automatic, Contours → 20, ColorFunction → "DarkRainbow"]
ContourPlot[u[π, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"x", "y"},
 FrameStyle → {Black, Black}, FrameTicksStyle → Directive["Label", 14],
 PlotRange → {-1, 1}, PlotLegends → Automatic, Contours → 20,
 ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
ContourPlot[u[3 * π / 2, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"x", "y"},
 FrameStyle → {Black, Black}, FrameTicksStyle → Directive["Label", 14],
 PlotRange → {-1, 1}, PlotLegends → Automatic, Contours → 20,
 ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
ContourPlot[u[2 * π, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"x", "y"},
 PlotLegends → Automatic, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 Contours → 20, ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
```

Out[373]=



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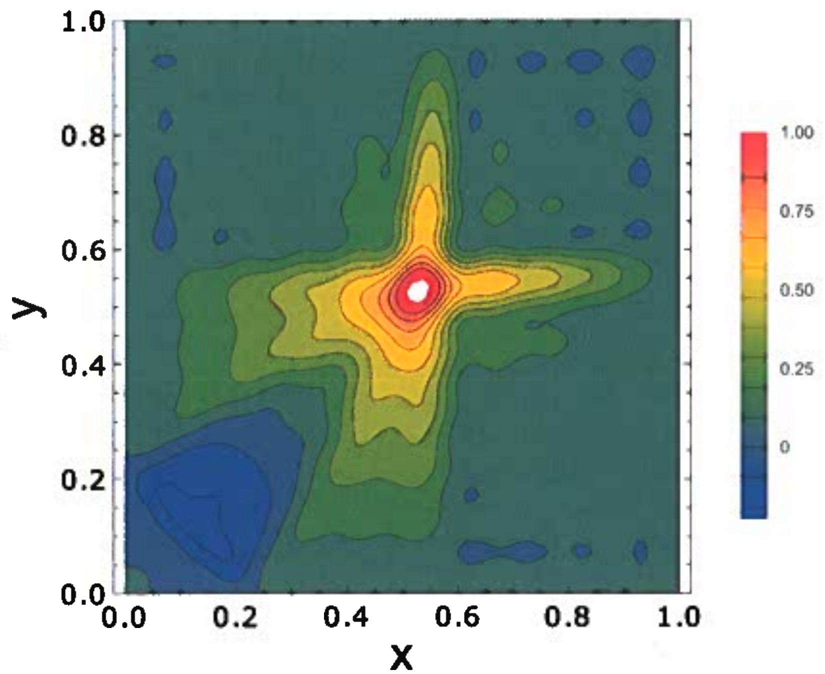

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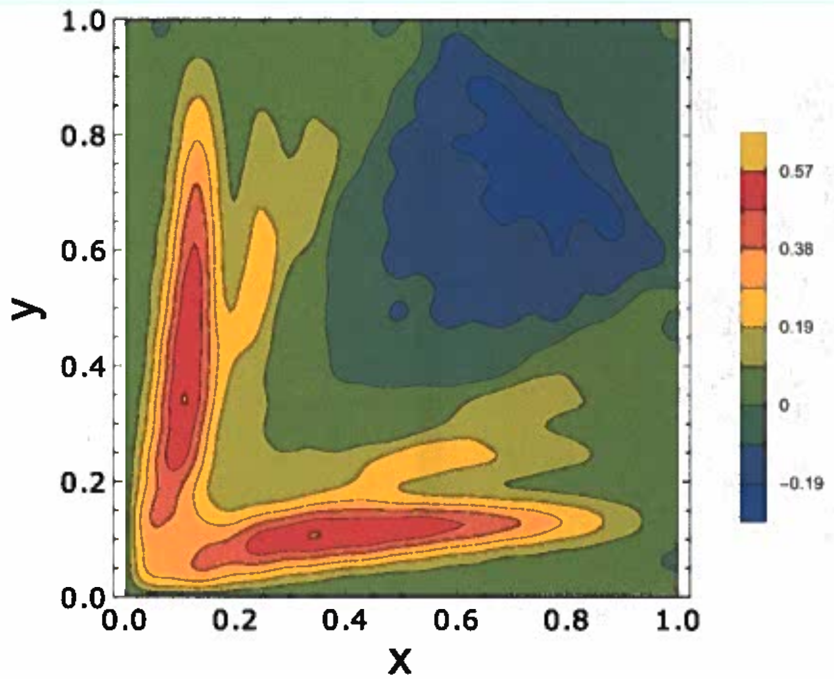

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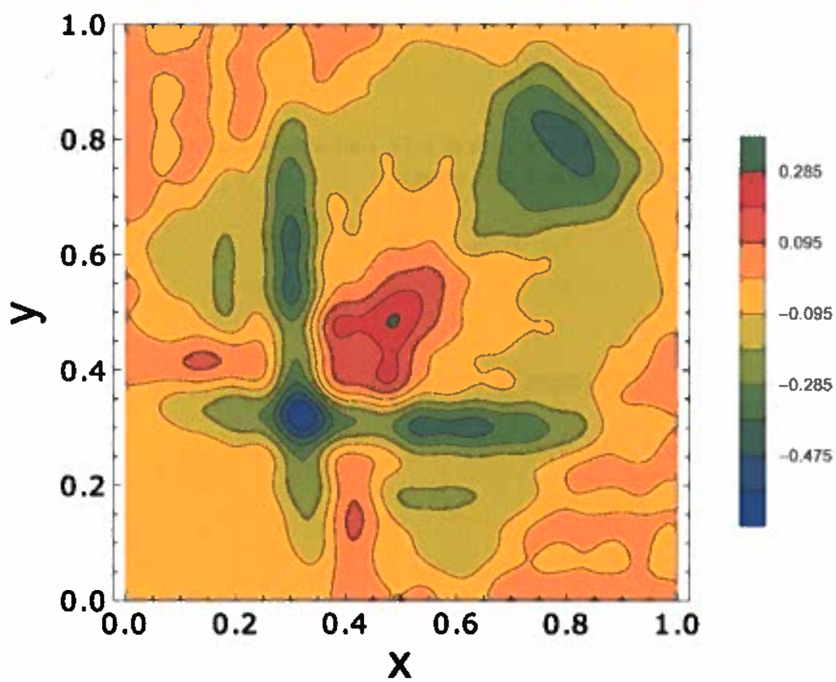

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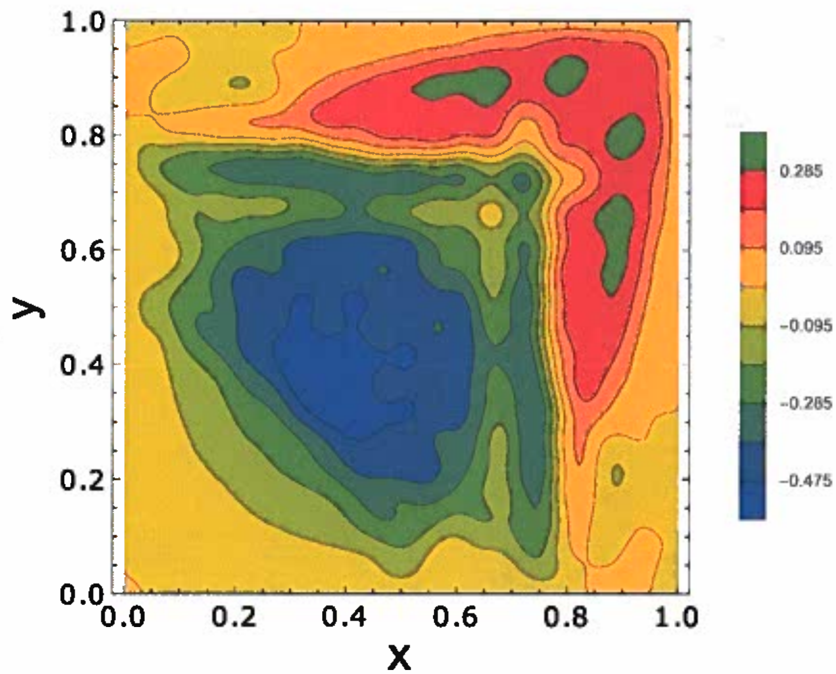


a=2 and b=2

In[378]:=
```
Simplify[
 Integrate[Integrate[Sin[n * π * x] * Sin[m * π * y] * Sin[2 * x] * Sin[2 * y], {x, 0, 1}],
  {y, 0, 1}], Assumptions → n ∈ Integers && m ∈ Integers]
```

Out[378]=

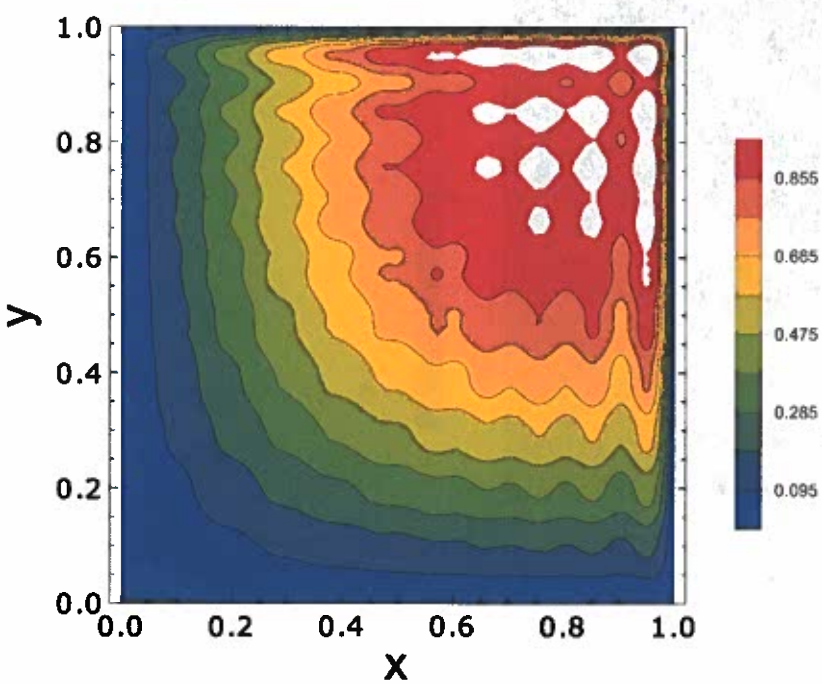
$$\frac{(-1)^{m+n}\, m\, n\, \pi^2\, \mathrm{Sin}[2]^2}{\left(-4 + m^2 \pi^2\right)\left(-4 + n^2 \pi^2\right)}$$

In[379]:=
```
u[t_, x_, y_] :=
  4 * Sum[Sum[((-1)^(m+n) m n π^2 Sin[2]^2)/((-4 + m^2 π^2) (-4 + n^2 π^2)) * Cos[Sqrt[n^2 + m^2] * π * t] * Sin[n * π * x] *
      Sin[m * π * y], {n, 1, 20}], {m, 1, 20}];
```

Snapshots in time

In[380]:=
```
ContourPlot[u[0, x, y], {x, 0, 1}, {y, 0, 1},
 FrameLabel → {"X", "Y"}, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 PlotLegends → Automatic, Contours → 20, ColorFunction → "DarkRainbow"]
ContourPlot[u[π / 2, x, y], {x, 0, 1}, {y, 0, 1},
 FrameLabel → {"X", "Y"}, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 PlotLegends → Automatic, Contours → 20, ColorFunction → "DarkRainbow"]
ContourPlot[u[π, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"X", "Y"},
 FrameStyle → {Black, Black}, FrameTicksStyle → Directive["Label", 14],
 PlotRange → {-1, 1}, PlotLegends → Automatic, Contours → 20,
 ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
ContourPlot[u[3 * π / 2, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"X", "Y"},
 FrameStyle → {Black, Black}, FrameTicksStyle → Directive["Label", 14],
 PlotRange → {-1, 1}, PlotLegends → Automatic, Contours → 20,
 ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
ContourPlot[u[2 * π, x, y], {x, 0, 1}, {y, 0, 1}, FrameLabel → {"X", "Y"},
 PlotLegends → Automatic, FrameStyle → {Black, Black},
 FrameTicksStyle → Directive["Label", 14], PlotRange → {-1, 1},
 Contours → 20, ClippingStyle → Automatic, ColorFunction → "DarkRainbow"]
```

Out[380]=



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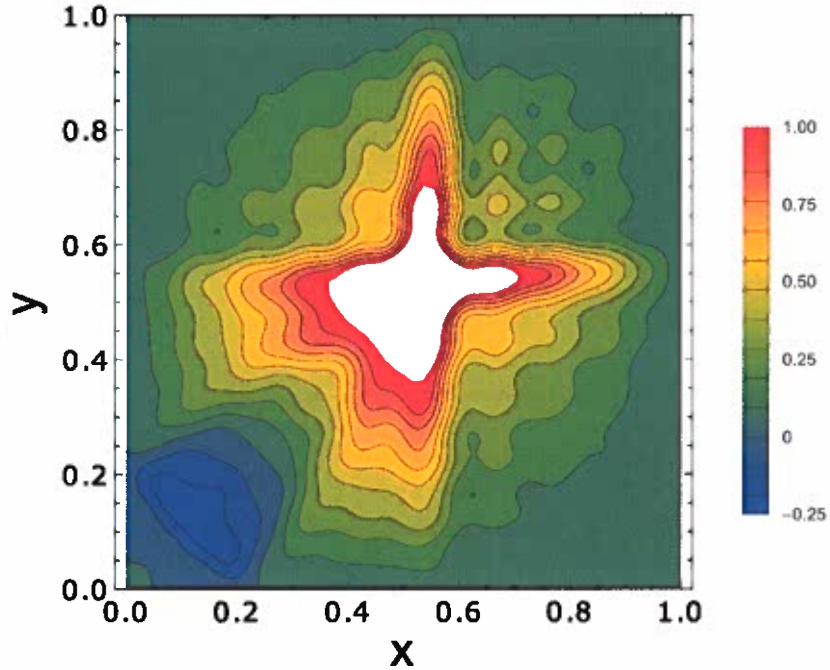
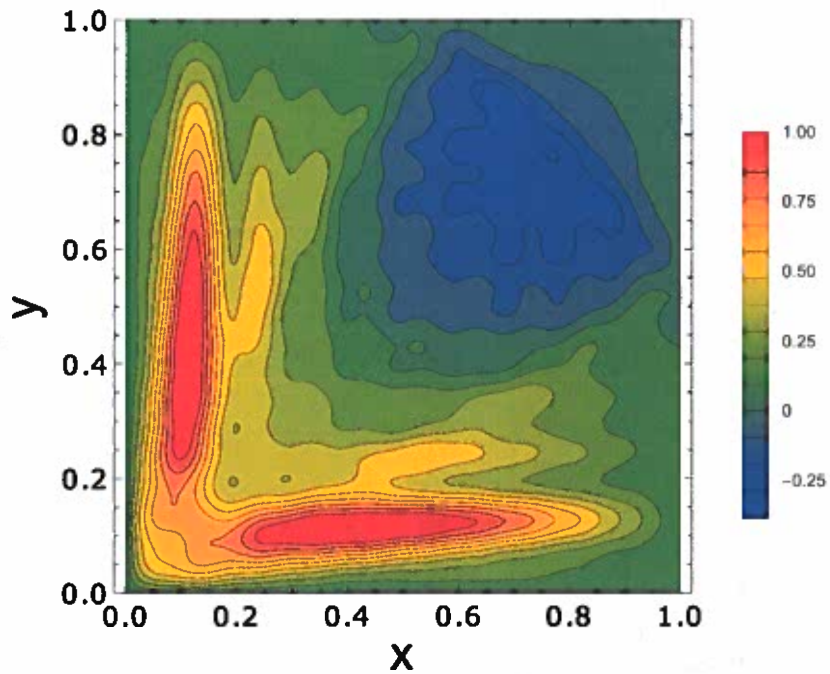


Out[382]=



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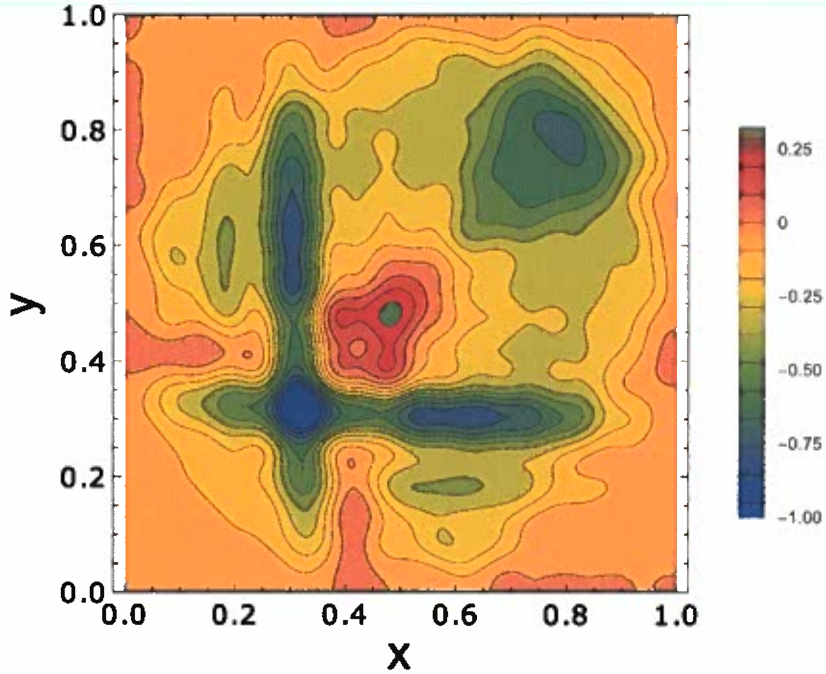

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