Lecture 1: What are Partial Differential Equations

Ordinary Differential Equations

Theorem - Consider the initial value problem

$$\frac{dx}{dt} = f(x,t) \quad (1)$$

$$x(0) = x_0 .$$

Suppose that f is differentiable and f' is continuous. Then (1) has a unique solution on some time interval $(-\tau, \tau)$.

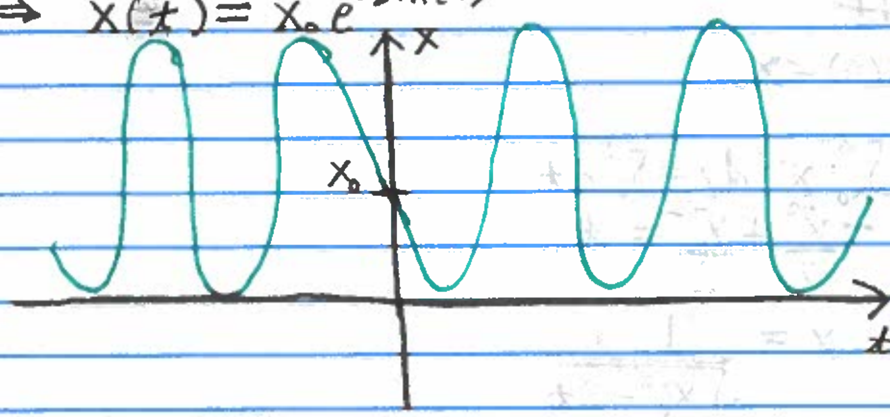
Example:

$$\frac{dx}{dt} = -\cos(t)x$$

$$\Rightarrow \int_{x_0}^{x} \frac{1}{x}\,dx = \int_0^t -\cos(t)\,dt$$

$$\Rightarrow \ln\left(\frac{x}{x_0}\right) = -\sin(t)$$

$$\Longrightarrow x(t) = x_0 e^{-\sin(t)}$$



Solution exists for all time.

Example:

$$\frac{dx}{dt} = x^{1/2}$$

$$x(0) = 0$$

$$\Rightarrow \int_0^x x^{-1/2}\,dx = \int_0^t dt$$

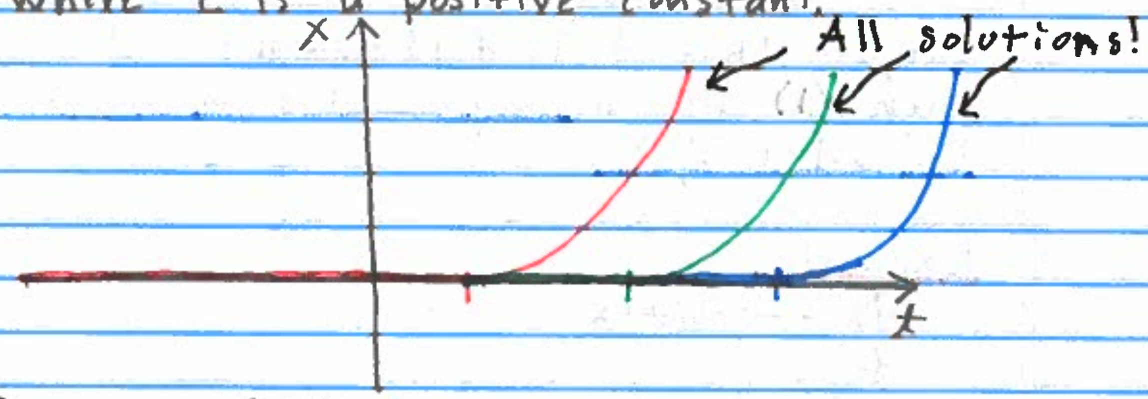
$$\Rightarrow 2x^{1/2} = t$$

$$\Rightarrow x(t) = \frac{t^2}{4}$$

There is, however, an infinite number of solutions

$$x(t) = \begin{cases} \frac{1}{4}(t-c)^2, & t \geq c \\ 0, & t < c \end{cases}$$

where c is a positive constant.



<u>Consequence:</u> This equation has non-unique solutions.

<u>Example:</u>

$$\frac{dx}{dt} = x^3$$

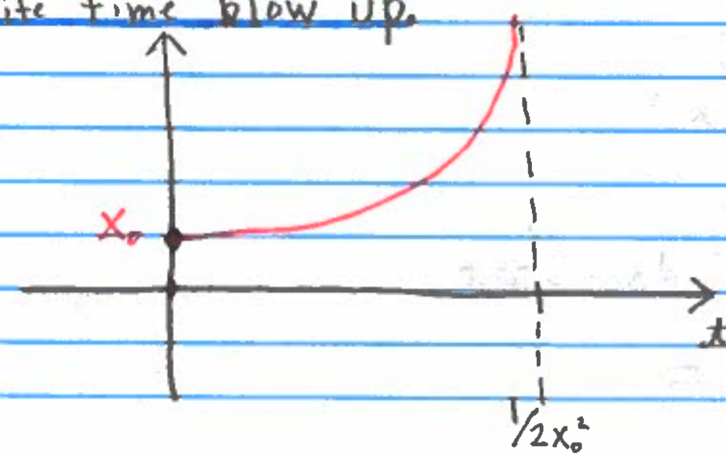
$$x(0) = x_0$$

$$\Rightarrow \int_{x_0}^{x} x^{-3}\,dx = \int_0^t dt$$

$$\Rightarrow -\frac{x^{-2}}{2} + \frac{x_0^{-2}}{2} = t$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{x_0^{-2} - 2t}}$$

The solution diverges at $t = 1/2x_0^2$! This is called <u>finite time blow up.</u>



Why Care?

The theory of differential equations is essentially completely resolved.

(i) It is easy to predict when solutions are unique.

(ii) Solutions, can blow up in finite time. Again this is easy to predict

(iii) Solutions can be run backwards in time.

(iv) There exist robust numerical solvers of ordinary differential equations.

* For partial differential equations there is no robust theory of existence, uniqueness, and regularity *

Partial Differential Equations

A differential equation is called a partial differential equation if the function depends on more than one variable.

Example:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{(Leibniz notation)}$$

u is a function of time t and a spatial coordinate x. Equivalent notation is

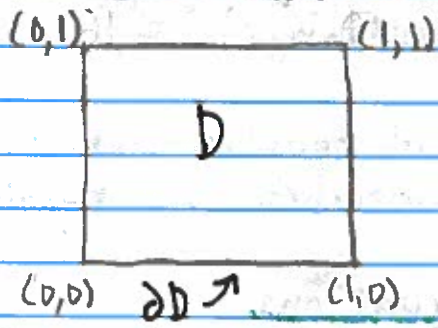
$$u_t = u_{xx} \qquad \text{(Subscript notation)}$$

*The order of a PDE is the highest order derivative in the equation. This equation is a second order linear equation.

Definition – A classical solution to a PDE of order n on a domain D is a $C^n(D)$ function satisfying the PDE on D. The domain D is a connected open set with smooth boundary ∂D.

Example:

$$U_t = U_{xx} + U_{yy}$$
$$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$



Solutions:

$$u(t,x,y) = t + \tfrac{1}{4}x^2 + \tfrac{1}{4}y^2$$
$$u(t,x,y) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-(x^2+y^2)}{4t}\right)$$
$$u(t,x,y) = e^{-t}\sin(x)\sin(y)$$

Linearity

A linear operator L is an operation on functions u, v satisfying

$$L[u+v] = L[u] + L[v],\quad L[cu] = cL[u],$$

where c is a constant.

Examples:

1. $L[u] = \dfrac{\partial u}{\partial x}$, Other notation: $L = \partial_x$.

2. $L[u] = \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial t}$, Other notation: $L = \partial_x - \partial_t$

3. $L[u] = \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}$, Other notation: $L = \partial_t - \partial_x^2 - \partial_y^2$.

Definition - A homogeneous linear differential equation has the form

$$L[u]=0$$

where L is a differential operator.

Example:

Consider the PDE

$$u_{tt}-K(x)u_t=u_{xx}$$

this is a homogeneous linear differential equation with operator

$$L=\partial_{tt}-K(x)\partial_t-\partial_{xx}$$

Principle of linear superposition.

Proposition - If u_1, u_2 solve $L[u]=0$, then u_1+u_2 and cu are also solutions

proof:

Let u_1, u_2 solve $L[u]=0$.

1. $L[u_1+u_2]=L[u_1]+L[u_2]=0+0=0$
2. $L[cu]=cL[u]=c\cdot 0=0$

Example:

$u_1(t,x)=(t+\frac{1}{2}x^2)$ and $u_2(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$

solve $u_t=u_{xx}$. Therefore,

$$u(t,x)=c_1(t+\tfrac{1}{2}x^2)+\frac{c_2}{\sqrt{4\pi t}}e^{-x^2/4t}$$

also solve $u_t=u_{xx}$.

Theorem - The set of all solutions to a linear differential equation forms a vector space.

Definition - An inhomogeneous linear differential equation has the form

$$L[u]=f.$$

Theorem - Let U^* be a particular solution to the inhomogeneous linear equation $L[U]=f$. The general solution is then U^*+U^H where U^H is a solution to the homogeneous equation.

proof:

1. $L[U^*+U^H]=L[U^*]+L[U^H]=f+0=f.$
2. Suppose $\bar{U}$ is another solution, i.e. $L[\bar{U}]=f.$
   Then,
   $$L[\bar{U}-U^*]=L[\bar{U}]-L[U^*]=0.$$
   $\Rightarrow$ $\bar{U}-U^*$ is a solution to the homogeneous equation.

■

Coordinate Transformations

How does a P.D.E. change under a coordinate transformation?

Convert $\frac{\partial}{\partial x}$ to polar coordinates $r=(x^2+y^2)^{1/2}$, $\theta=\tan^{-1}(y/x)$:

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$
$$=\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r}-\frac{y}{x^2+y^2}\frac{\partial}{\partial \theta}$$
$$=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Convert $\frac{\partial^2}{\partial x^2}$ to polar coordinates:

$$\frac{\partial^2}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial}{\partial x}$$
$$=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$
$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r}\right)+\cos\theta\frac{\partial}{\partial r}\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$
$$-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial r}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$\Longrightarrow \frac{\partial^2}{\partial x^2} = \cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$