Lecture 10: Numerical Solutions to the Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$u(t,0) = \alpha(t)$

$u(t,\ell) = \beta(t)$

$u(0,x) = f(x)$

$u_t(0,x) = g(x)$

Discretization:

$t_j = j\Delta t \quad x_i = i\Delta x$

$$U_{tt}(t_j, x_i) = \frac{U_{j+1,i} - 2U_{j,i} + U_{j-1,i}}{\Delta t^2}$$

$$U_{xx}(t_j, x_i) = \frac{U_{j,i+1} - 2U_{j,i} + U_{j,i-1}}{\Delta x^2}$$

$$\Rightarrow \boxed{U_{j+1,i} = \frac{\Delta t^2}{c^2 \Delta x^2}(U_{j,i+1} - 2U_{j,i} + U_{j,i-1}) + 2U_{j,i} - U_{j-1,i}}$$

The CFL condition is

$$\boxed{\frac{\Delta t}{c\Delta x} < 1.}$$

Intialization:

The above scheme does not work when $j=1$.

One idea is to use the derivative to approximate $U_{2,i}$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{U_{2,i} - U_{1,i}}{\Delta t} + O(\Delta t) = g(x_i)$$

$$\Rightarrow U_{2,i} = \Delta t\, g(x_i) + U_{1,i}$$

However, this is a $O(\Delta t)$ approximation, not $O(\Delta t^2)$

Alternative:

$$\frac{u(\Delta t, x_i) - u(0, x_i)}{\Delta t} = \frac{\partial u}{\partial t}(0, x_i) + \frac{1}{2}\frac{\partial^2 u}{\partial t^2}(0, x_i)\Delta t + \mathcal{O}(\Delta t^2)$$

$$= g(x_m) + \frac{c^2}{2}\frac{\partial^2 u}{\partial x^2}(0, x_i)\Delta t + \mathcal{O}(\Delta t^2)$$

$$= g(x_m) + \frac{c^2}{2}\cancel{f}f''(x_i)\Delta t + \mathcal{O}(\Delta t^2).$$

$$\Longrightarrow \boxed{U_{2,i} = U_{1,i} + \Delta t\, g(x_m) + \frac{c^2 \Delta t^2}{2} f''(x_i).}$$

In vector notation

$$\vec{U}_1 = f(x)$$

$$\vec{U}_2 = \vec{U}_1 + \Delta t\, g(x) + \frac{c^2\Delta t^2}{2} f''(x)$$

$$\vec{U}_{j+1} = \frac{\Delta t^2}{c^2 \Delta x^2} D\vec{U}_j + 2\vec{U}_j - \vec{U}_{j-1}$$

$$D = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$