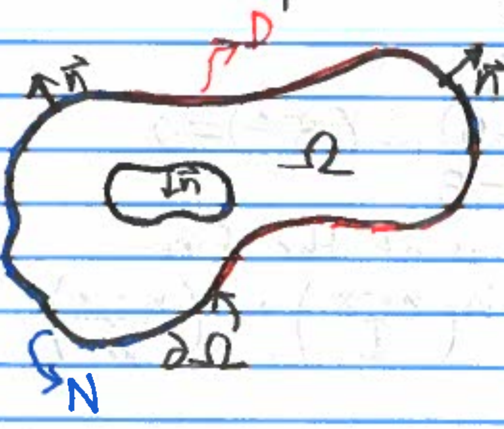
Lecture 11: Laplace's Equation



$U_t = U_{xx} + U_{yy}$

$U|_D = f(x,y)$ (Dirichlet)

$\nabla U \cdot \vec{n} = 0$ (Neumann)

$U(0,x,y) = g(x,y)$ (Initial Condition)

Steady State Distribution satisfies:

$$U_{xx} + U_{yy} = \nabla \cdot \nabla U = \Delta U = 0$$

Example:

$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le b\}$.

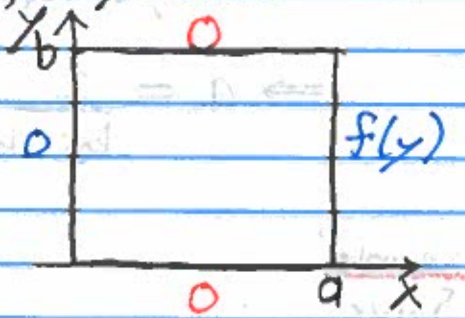
$\Delta u = 0$

$u(0,y) = 0$

$u(a,y) = f(y)$

$u(x,0) = 0$

$u(x,b) = 0$



$u(x,y) = X \cdot Y$

$\Rightarrow X''Y + XY'' = 0$

$\Rightarrow -\dfrac{X''}{X} = \dfrac{Y''}{Y} = \lambda$

$\Rightarrow Y'' = \lambda Y$

$u(x,0) = 0 \Rightarrow Y(0) = 0$

$u(x,b) = 0 \Rightarrow Y(b) = 0$

Solutions are only possible if $\lambda = -\omega^2 < 0$.

$\Rightarrow Y = A\sin\left(\dfrac{n\pi y}{b}\right), \quad \omega = \dfrac{n\pi}{b}$

$X'' = -\lambda X$

$\Rightarrow X'' = w^2 X$

$\Rightarrow X = Ae^{wx} + Be^{-wx}$

$u(0,y) = 0 \Rightarrow (A+B)\sin\left(\frac{n\pi y}{b}\right) = 0$

$\Rightarrow A = -B$

A generic solution is of the form:

$$u_n(x,y) = a_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow u(a,y) = f(y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy = \frac{b}{2} a_n \sinh\left(\frac{n\pi a}{b}\right)$$

$$\Rightarrow a_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy.$$

Example:

Solve

$\Delta u = 0$

$u(0,y) = 1$

$u(\pi,y) = 1$

$u(x,0) = 1$

$u(x,\pi) = \cos(2x).$

$\tilde{u}(x,y) = 1$ solves Laplace's equation. Define

$v(x,y) = u(x,y) - 1$

$\Rightarrow \Delta v = 0$

$v(0,y) = v(\pi,y) = v(x,0) = 0$

$v(x,\pi) = \cos(2x) - 1$

$$\Rightarrow v(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx)\sinh(ny)$$

$$\Rightarrow a_n = \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} \sin(nx)(\cos(2x)-1)\,dx$$

$$\Rightarrow a_n = \frac{4(1-(-1)^n)}{n^3-4n} \cdot \frac{2}{\pi \sinh(n\pi)}$$

$$\Rightarrow u(t,x) = 1 + \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \frac{8(1-(-1)^n)}{\pi \sinh(n\pi)(n^3-4n)} \sin(nx)\sinh(ny).$$

Maximum Principle: - If u solves Laplace's equation then u obtains it maximum and minimum on $\partial\Omega$.

proof (2-D):

If u obtains max (min) in Ω at (a,b) then $\nabla u|_{(a,b)} = 0$. The discriminant is

$$u_{xx}u_{yy} - u_{xy}^2 = \det\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \lambda_1 \cdot \lambda_2 \rightarrow \text{product of eigenvalues}$$

$$u_{xx}+u_{yy} = \mathrm{tr}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = \lambda_1 + \lambda_2 = 0$$

$\Rightarrow \lambda_1 = -\lambda_2$

$\Rightarrow$ discriminant < 0

$\Rightarrow$ saddle point (cannot be a max(min)).

(More technical proof needed, but this is the idea of the proof)

Theorem - Solutions to Laplace's equation with Dirichlet boundary conditions are unique.

proof:

Suppose u_1, u_2 solve Laplace's equation. Then, $v = u_1 - u_2$ solves

$$\Delta v = 0$$
$$v|_{\partial\Omega} = 0$$

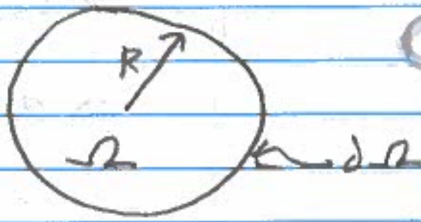
The maximum principle implies $v \equiv 0$. ∎

__Example:__

$0 = \Delta u$

$\Omega = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq R^2\}$

$u(R,\theta) = f(\theta)$



Convert to polar coordinates:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

$$u(R,\theta) = f(\theta)$$

$$u(r,0) = u(r,2\pi)$$

$$\Rightarrow u_\theta(r,0) = u_\theta(r,2\pi)$$

Assume

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Rightarrow \Theta R'' + \frac{1}{r} R' \Theta + \frac{1}{r^2} R \Theta'' = 0$$

$$\Rightarrow \frac{R'' + \frac{1}{r} R'}{R} r^2 = -\frac{\Theta''}{\Theta} = \lambda$$

$$\Rightarrow \Theta'' = -\lambda \Theta, \quad r^2 R'' + r R' = \lambda R$$

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta)$$

$$\Theta(0) = \Theta(2\pi) \Rightarrow A\cos(2\pi\sqrt{\lambda}) + B\sin(2\pi\sqrt{\lambda}) = A$$

$$\Theta'(0) = \Theta'(2\pi) \Rightarrow -A\sin(2\pi\sqrt{\lambda}) + B\cos(2\pi\sqrt{\lambda}) = B$$

$$\Rightarrow \underbrace{\begin{bmatrix} \cos(2\pi\sqrt{\lambda}) & \sin(2\pi\sqrt{\lambda}) \\ -\sin(2\pi\sqrt{\lambda}) & \cos(2\pi\sqrt{\lambda}) \end{bmatrix}}_{= I \text{ (solvability condition).}} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}$$

$$\Rightarrow \lambda = n^2$$

We now solve for $R(r)$:

$$r^2R'' + rR' - \lambda R = 0$$

Guess $R = ar^m$

$$\Rightarrow a\cdot m(m-1)r^m + amr^m - an^2r^m = 0$$
$$\Rightarrow m^2 = n^2$$
$$\Rightarrow m = \pm n$$
$$\Rightarrow R = ar^n + br^{-n}$$

However, at 0 $br^{-n} = 0 \Rightarrow b = 0$.

The Generic Solution is given by:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n r^n \sin(n\theta)$$

At $r=R$:

$$u(R,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n R^n \cos(n\theta) + \sum_{n=1}^{\infty} b_n R^n \sin(n\theta)$$

$$\Rightarrow a_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\,d\theta$$
$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$
$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

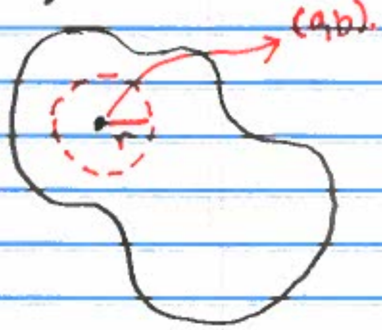
Theorem (Mean Value Theorem) – If u satisfies $\Delta u = 0$ then:

$$\underbrace{u(a,b)}_{\text{value of } u} = \frac{1}{2\pi} \underbrace{\int_0^{2\pi} u|_{B((a,b),R)}\,d\theta}_{\text{Integral over ball of radius } R.}$$

~~proof~~:

Follows from previous solution by solving $\Delta u = 0$ subject to its own boundary conditions and noting that for polar coordinates (R,θ) about (a,b).

$$u(a,b) = \frac{a_0}{2} = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\,d\theta.$$



*Laplace's equation averages out heat, charge, etc!

*We can also integrate with respect to r:

$$\int_0^R u(a,b)\,R\,dR = \frac{1}{2\pi}\int_0^{2\pi}\int_0^R u|_{B((a,b),R)}\,R\,dR\,d\theta$$

$$\Rightarrow u(a,b)\frac{R^2}{2} = \frac{1}{2\pi}\text{ Total heat in ball of radius } R.$$

$$\Rightarrow \boxed{u(a,b) = \frac{\text{Total heat in Ball}}{\text{Area of Ball}}}$$

Example:

$\Delta u = 0$

$\Omega = \{(x,y)\in\mathbb{R}^2 ; x^2+y^2\le R^2\}$

$u(R,\theta) = \sin^2\theta$

$$\Rightarrow u(r,\theta) = \frac{1}{2} - \frac{r^2}{2}\cos(2\theta).$$