Lecture 12: Distributions (Generalized Functions)

Solving Linear Equations

Let $A \in \mathbb{R}^{n\times n}$. Solve

$$Ax = b$$

$$\Rightarrow x = A^{-1}b$$

The solution exists and is unique if A is nonsingular.

Example:

Find the inverse of

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}.$$

Augmented matrix:

$$\left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right]$$

Row Reduce:

$$A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$

Why does this work???

Inverse of a matrix (abtract)

How does a matrix work. Let $\vec{x} \in \mathbb{R}^n$ such that

$$\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n, \quad e_i = [0, 0, \dots 0, \underset{\substack{\uparrow \\ i\text{-th slot}}}{1}, 0, \dots 0]$$

$$\Rightarrow A\vec{x} = x_1 A\vec{e}_1 + \dots x_n A\vec{e}_n$$

$$= x_1\vec{c}_1 + \dots x_n\vec{c}_n.$$

$\vec{c}_1, \dots, \vec{c}_n$ are the columns of A.

To find the columns of A, need to compute Ae_i.

To find the columns of A^{-1} need to figure out

$$A^{-1}\vec{e}_i = \vec{c}_i$$

unknown

$$\Rightarrow \vec{e}_i = A\vec{c}_i$$ (Solve this equation for c_i)

The construction of the inverse is equivalent to solving an equation!!

Linear Operators

Solve $\mathcal{L}[f] = b.$

$$\Rightarrow f = \mathcal{L}^{-1}[b].$$

To find $\mathcal{L}^{-1}$ need to compute

$$\mathcal{L}^{-1}[\text{basis vector}] = \text{column vector}$$

$$\Rightarrow \text{basis vector} = \mathcal{L}[\text{column vector}]$$

What is the basis vector??

What do we mean by column vector??

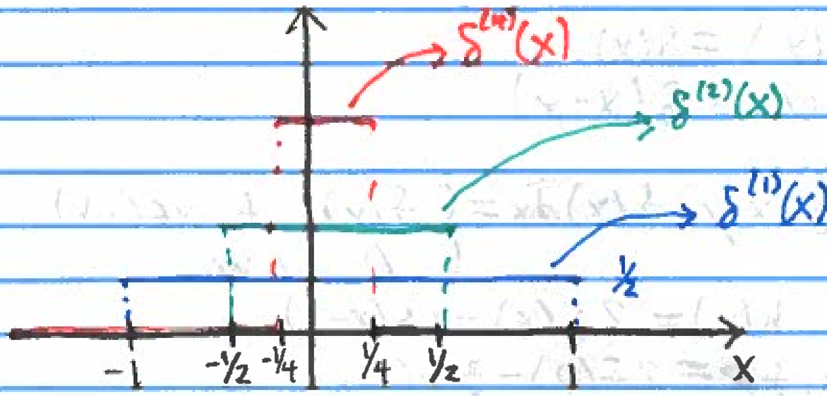
Operators on Function Spaces

	$\mathcal{L} = \frac{d^2}{dx^2}$	$A \in \mathbb{R}^{n\times n}$
Domain	C^2	$\mathbb{R}^n$
Range	C^0	$\mathbb{R}^n$
Basis vector	function?	$\vec{e}_i$ (vector)
Columns	function?	$\vec{c}_i$ (vector)

	$f: \mathbb{R} \to \mathbb{R}$	$\vec{v}: \mathbb{N} \to \mathbb{R}$
Domain	$\mathbb{R}$	$\mathbb{N}$
Range	$\mathbb{R}$	$\mathbb{R}$
Basis vector	$\delta_x(f) = f(x)$	$e_i(\vec{v}) = v_i$
	$\langle \delta_x, f\rangle = f(x)$	$\Rightarrow \underbrace{\langle e_i, \vec{v}\rangle}_{\text{Inner product.}} = v_i$

Delta Sequence

$$\delta^{(n)}(x) = \begin{cases} n/2, & -1/n < x < 1/n \\ 0, & \text{o.w.} \end{cases}$$



1. $$\lim_{n\to\infty} \delta^{(n)}(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$

2. $$\int_{-\infty}^{\infty} \delta^{(n)}(x)\,dx = 1$$

$$\Rightarrow \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta^{(n)}(x)\,dx = 1.$$

3. $$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta^{(n)}(x) f(x)\,dx = f(0).$$

~~proof~~:

(1-2) are obvious. To prove property 3 note that

$$f(0) = \int_{-\infty}^{\infty} \delta^{(n)}(x) f(0)\,dx$$

$$\Rightarrow \left| f(0) - \int_{-\infty}^{\infty} \delta^{(n)}(x) f(x)\,dx \right| \le \int_{-1/n}^{1/n} \left| n/2 \, (f(0) - f(x)) \right| dx$$

$$\le \int_{-1/n}^{1/n} M\,dx$$

$$= \frac{2M}{n},$$

where $M = \max|f'(x)|$.

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Calculus of Delta Function.

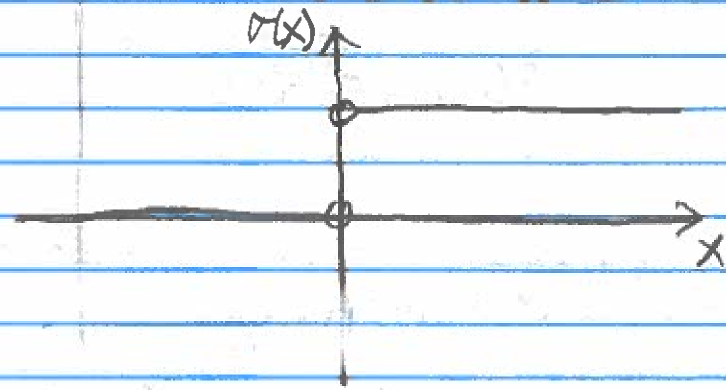
1. $\delta(x)$ is understood as a limit;

$$\delta_0(x) \text{"="} \lim_{n\to\infty} \delta^{(n)}(x).$$

2. $\delta_0(x) = \delta(x)$
3. $\delta_y(x) = \delta(x-y)$
4. $\int_a^b \delta(x-y) f(x)\,dx = \begin{cases} f(y) & \text{if } x\in(a,b) \\ 0 & \text{o.w.} \end{cases}$
5. Let $h(x) = 2\delta(x) - 3\delta(x-1)$
6. $\langle h, f\rangle = 2f(0) - 3f(1)$.
7. $x^3\delta(x) = 0$ since $\langle x^3\delta(x), f(x)\rangle = \langle \delta(x), x^3 f(x)\rangle$

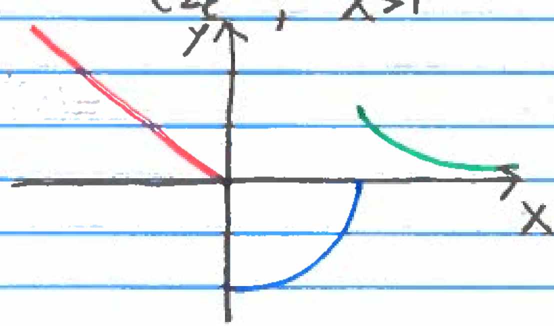
$$= 0^3\cdot f(0)$$
$$= 0.$$

8. $\int_{-1}^{x} \delta(y)\,dx = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases} = \sigma(x)$



Example:

$$f(x) = \begin{cases} -x, & x<0 \\ x^2-1, & 0<x<1 \\ 2e^{-x}, & x>1 \end{cases} \Rightarrow f'(x) = -\delta(x) + \frac{2}{e}\delta(x-1) + \begin{cases} -1, & x<0 \\ 2x, & 0<x<1 \\ -2e^{-x}, & x>1. \end{cases}$$



Theorem - If $f \in C^1$ then for all x

$$f(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} f(y)\,dy.$$

proof:

First, note that

$$\frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} f(x)\,dy = f(x).$$

Therefore,

$$\left| \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} f(y)\,dy - f(x) \right| = \frac{1}{\Delta x} \left| \int_{x-\Delta x/2}^{x+\Delta x/2} (f(y) - f(x))\,dy \right|$$

Let $m = \min|f'(x)|$, $M = \max|f'(x)|$. By the mean value theorem

$$\frac{f(y)-f(x)}{y-x} = f'(c)$$

$$\Rightarrow \frac{1}{\Delta x} \left| \int_{x-\Delta x/2}^{x+\Delta x/2} f(y) - f(x)\,dy \right| \le \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} |f(y) - f(x)|\,dy$$

$$\le \frac{1}{\Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} M \Delta x\,dy$$

$$= M \Delta x$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(y)\,dy = f(x).$$

$\Rightarrow$ We can think of functions as averaging.

Construction of a Basis Vector on functions.

Properties:

1. $\langle \delta_y, f \rangle = f(y)$

$\Longrightarrow \int_{-\infty}^{\infty} \delta_y(x) f(x)\,dx = f(y)$

Lets figure out what $\delta_y(x)$ is.

$$\delta_y(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\Delta x/2} \delta_y(z)\,dz$$

Let $h_{\Delta x}(z) = \begin{cases} 1 & \text{if } x-\frac{\Delta x}{2} < z < x+\Delta x/2 \\ 0 & \text{o.w.} \end{cases}$

$$\delta_y(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{-\infty}^{\infty} \delta_y(z)\, h_{\Delta x}(z)\,dz$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} h_{\Delta x}(y)$$

$$= \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{if } x = y \end{cases}$$

However, $\int_{-\infty}^{\infty} \delta_y(x)\,dx = \int_{-\infty}^{\infty} \delta_y(x) \cdot 1\,dx = 1.$

Formally, the δ_y-function satisfies

$$\boxed{\begin{array}{l} 1.\ \delta_y(x) = \begin{cases} 0 & \text{if } x \neq y \\ \infty & \text{if } x = y \end{cases} \\ 2.\ \int_{-\infty}^{\infty} \delta_y(x)\,dx = 1 \end{array}}$$

* However, no function exists that satisfies these properties!!!

Example:

Simplify the following:

1. $f(x)\delta(x-a) = f(a)\delta(x-a)$

2. $f(x)\delta(1-x^2)$

$$\langle f(x)\delta(1-x^2), g(x)\rangle = \int_{-\infty}^{\infty} f(x)g(x)\delta(1-x^2)\,dx$$

$$= \int_{-\infty}^{0} f(x)g(x)\delta(1-x^2)\,dx + \int_{0}^{\infty} f(x)g(x)\delta(1-x^2)$$

Let $u = 1-x^2$, $v = 1-x^2 \Rightarrow x = -\sqrt{1-u}$, $x = \sqrt{1-u}$

$$\Rightarrow \langle f(x), \delta(1-x^2)\rangle = -\int_{-\infty}^{1} \frac{f(-\sqrt{1-u})\,g(-\sqrt{1-u})}{2\cdot x}\delta(u)\,du$$

$$-\int_{-\infty}^{1} \frac{f(\sqrt{1-v})\,g(\sqrt{1-u})}{2x}\delta(v)\,du$$

$$= -\frac{f(-1)g(-1)}{2} - \frac{f(1)g(1)}{2}$$