Lecture 13: Green's Functions.

Recall to solve the equation

$$Ax=b$$

we invert:

$$x=A^{-1}b.$$

$$\Rightarrow x=b_1A^{-1}e_1+b_2A^{-1}e_2+\dots+b_nA^{-1}e_n=\sum_{i=1}^{n}(A^{-1}e_i)b_i$$

We just need to find $A^{-1}e_i$ which is equivalent to solving the equations:

$$Aa_1=e_1$$
$$Aa_2=e_2 \quad \Rightarrow a_i = \text{column } i \text{ of matrix } i.$$
$$\vdots$$
$$Aa_n=e_n$$

Example:

Solve the following equation:

$$\begin{cases}-u''(x)=f(x)\\ u(0)=u(1)=0\end{cases}$$

Let $\mathcal{L}=\frac{d^2}{dx^2} \Rightarrow u(x)=\mathcal{L}^{-1}f(x)$. We need to find $\mathcal{L}^{-1}$.

Idea: Solve

$$-u''(x)=\underbrace{\delta(x-y)}_{\text{basis function centred at } y.}$$

$\Rightarrow$ 1. $u''(x)=0,\ x\neq y$ (Properties of δ-function)

2. $\lim_{x\to y^+}u'(x)-\lim_{x\to y^-}u'(x)=-1$ (Jump Condition)

3. $\lim_{x\to y^+}u(x)=\lim_{x\to y^-}u(x)$ (Continuity).

4. $u(0)=u(1)=0$ (Boundary Conditions)

The generic solution is of the form:

$$u(x)=\begin{cases}Ax+B, & x<y\\ Cx+D, & x>y\end{cases}$$ (Satisfies Property 1.)

Property 4 implies:

$$B=0$$
$$C+D=0 \Rightarrow D=-C.$$

Property 2 implies:

$$A-C=1$$
$$\Rightarrow C=A-1$$

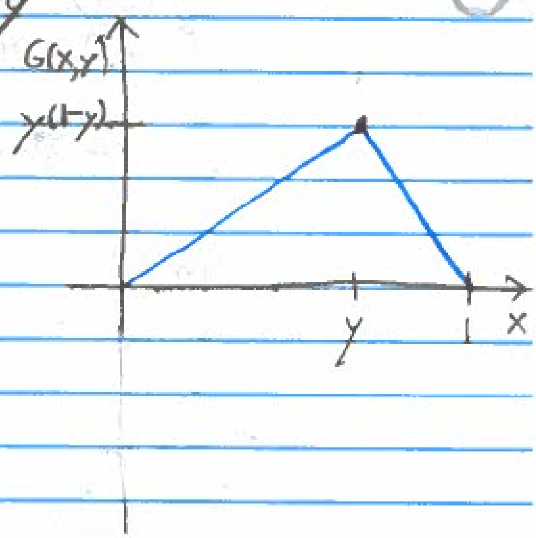
Property 3 implies:

$$Ay+B=Cy+D$$
$$\Rightarrow Ay=(A-1)y+1-A$$
$$\Rightarrow Ay=Ay-y+1-A$$
$$\Rightarrow A=1-y.$$

The Green's function is this solution:

$$G(x,y)=\begin{cases} x(1-y), & x<y \\ -yx+1-(1-y) & x>y \end{cases}$$

$$\Rightarrow G(x,y)=\begin{cases} x(1-y), & x<y \\ y(1-x), & x>y \end{cases}$$



The solution is therefore:

$$u(x)=\int_0^1 \underbrace{G(x,y)}_{\text{y-column of inverse}}\underbrace{f(y)}_{\text{y-component of function}}dy$$

$\Rightarrow$ The inverse of the operator

$$\mathcal{L}[u]=u'', \quad u(0)=u(1)=0$$

is

$$\mathcal{L}^{-1}[f]=\int_0^1 G(x,y)f(y)\,dy.$$

Example:

Solve the boundary value problem

$$-u'' + u = f(x)$$
$$u(0) = u(1) = 0$$

Find the Green's Function:

$$u'' - u = \delta(x-y)$$

1. Solve $-u'' + u = 0$

$$u(0) = u(1) = 0$$
$$\Rightarrow u(x,y) = \begin{cases} A\sinh(x), & x \leq y \\ B\sinh(1-x), & x \geq y \end{cases}$$

2. Jump Condition:

$$A\cosh(y) + B\cosh(1-y) = 1$$

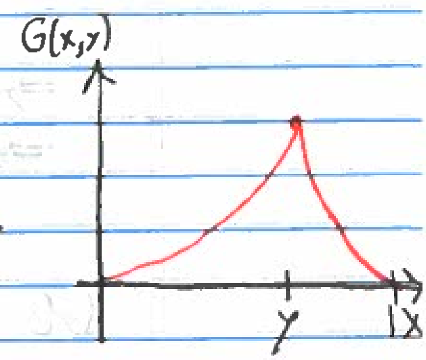
3. Continuity:

$$A\sinh(y) = B\sinh(1-y)$$
$$\Rightarrow B = \frac{A\sinh(y)}{\cancel{B}\sinh(1-y)}$$

$$\Rightarrow A\cosh(y) + A\sinh(y)\coth(1-y) = 1$$
$$\Rightarrow A(\sinh(1-y)\cosh(y) + A\sinh(y)\cosh(1-y)) = \sinh(1-y)$$
$$\Rightarrow A\sinh(1-y+y) = \sinh(1-y)$$
$$\Rightarrow A = \frac{\sinh(1-y)}{\sinh(1)}$$

Therefore,

$$G(x,y) = \begin{cases} \dfrac{\sinh(1-y)\sinh(x)}{\sinh(1)}, & x \leq y \\ \dfrac{\sinh(y)\sinh(1-x)}{\sinh(1)}, & x > y. \end{cases}$$



Generic Problem:

$$p(x)\frac{d^2u}{dx^2}+q(x)\frac{du}{dx}+r(x)u(x)=f(x)$$

Properties of $G(x,y)$:

1. Solves the homogeneous equation for all $x \neq y$.
2. Satisfies boundary conditions
3. Is continuous
4. For each y $\frac{\partial G}{\partial x}$ has a jump discontinuity of magnitude $1/p(y)$ at $x=y$.

Example:

Find the Green's function for

$$u''(x)-u(x)=f(x)$$

$$u(0)=0, \quad \lim_{x\to\infty} u(x)=0.$$

1. $G(x,y)=\begin{cases} A\sinh(x), & x<y \\ Be^{-x} & x>y \end{cases}$

2. Continuity:

$$A\sinh(y)=Be^{-y}$$

$$\Rightarrow B=Ae^{y}\sinh(y)$$

3. Jump Condition:

$$-Be^{-y}-A\cosh(y)=1$$

$$\Rightarrow A(\sinh(y)+\cosh(y))=-1$$

$$\Rightarrow A=\frac{-1}{\sinh(y)+\cosh(y)}$$

$$G(x,y)=\begin{cases} \dfrac{-\sinh(x)}{\sinh(y)+\cosh(y)}, & x<y \\[2ex] \dfrac{-\sinh(y)}{\sinh(y)+\cosh(y)}e^{y-x}, & x>y. \end{cases}$$

Check of Example 1:

Claim:

$$u(x)=\int_0^1 G(x,y)f(y)\,dy\,,$$

$$G(x,y)=\begin{cases} x(1-y)\,, & x<y\,,\\ y(1-x)\,, & x\geq y\end{cases}$$

Solves

$$-u''=f(x)\,,\quad u(0)=u(1)=0.$$

Check:

$$u(x)=\int_0^x y(1-x)f(y)\,dy+\int_x^1 x(1-y)f(y)\,dy$$

Recall: $$\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,dy=\frac{d}{dx}\left[H(x,b(x))-H(x,a(x))\right]$$

$$=\left.\frac{\partial H}{\partial x}\right|_{y=b(x)}+\left.\frac{\partial H}{\partial y}\right|_{y=b(x)}\frac{db}{dx}-\left.\frac{\partial H}{\partial x}\right|_{y=a(x)}-\left.\frac{\partial H}{\partial y}\right|_{y=a(x)}\frac{da}{dx}$$

$$=\int_{a(x)}^{b(x)}\frac{\partial h}{\partial x}\,dx\,dy+h(x,b(x))\frac{db}{dx}-h(x,a(x))\frac{da}{dx}.$$

$$\Rightarrow \frac{du}{dx}=\int_0^x -yf(y)\,dy+x(1-x)f(x)+\int_x^1(1-y)f(y)\,dy-x(1-x)f(x)$$

$$=\int_0^1 -yf(y)\,dy+\int_x^1 f(y)\,dy$$

$$\Rightarrow \frac{d^2u}{dx^2}=-f(x).$$